Parametric equation and Polar Coordinates:

Parametric equations

Def: If x and y are given as functions $x = f(t)$; $y = g(t)$, over an interval I of t-values, then the set of points $(x, y) = (f(t),\ g(t))$ defined by these equations is a parametric curve. The equations are parametric equations for the curve.

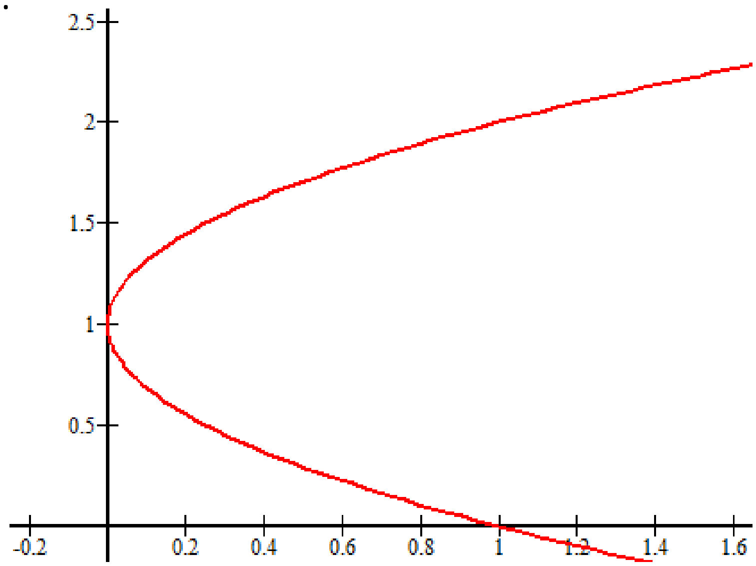
Notice: The variable t is a parameter for the curve, and it's domain I is the parameter interval. If I is a closed interval $a \le t \le b$, the point $(f(a), g(a))$ is the initial point of the curve and $(f(b), g(b))$ is the terminal point.

Ex: Sketch the curve defined by parametric equation.

$x = t^2 , t = t + 1 , -\infty < t < \infty.$

Sol:

t	x	y
-3	9	-2
-2	4	-1
-1	1	0
0	0	1
1	1	2
2	4	3
3	9	4



Ex: Graph the parametric curves:

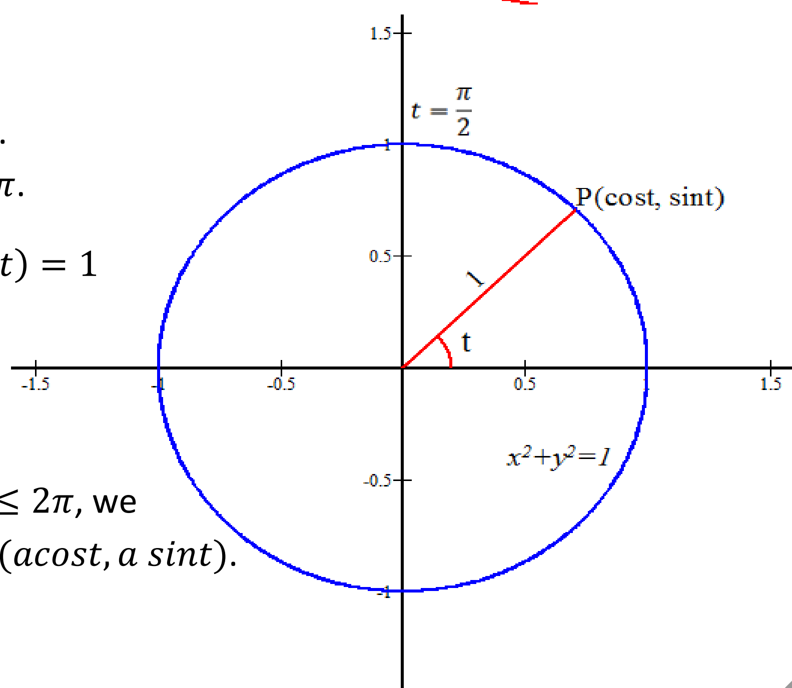
1) $x = cos(t), y = \sin(t)$, $0 \le t \le 2\pi$.
2) $x = acos(t) , y = asin(t) , 0 \le t \le 2\pi$.

Sol: 1) Since $x^2 + y^2 = cos^2(t) + sin^2(t) = 1$

as t increases from 0 to 2π.

The point $(x, y) = (cost, sint)$.

2) For $x = acos(t) , y = asin(t) , 0 \le t \le 2\pi$, we have $x^2 + y^2 = a^2, r = a$ and $(x, y) = (acost, a\ sint)$.



Ex: Graph the parametric curve $x = \sqrt{t}\,, y = t\,,\ t \geq 0$. (H.W)

Ex: Find a parameterization for the line through the point (a, b) having sloping m.

Sol: A Cartesian equation of the line is $y - b = m(x - a)$.

Let $t = x - a \Rightarrow x = a + t\ \&\ y = b + mt\,, -\infty < t < \infty$.

Ex: Sketch and identify the path traced by the point $P(x, y)$ if $x = t + \frac{1}{t}\,, y = t - \frac{1}{t}$

$t > 0$.(H.W) $[Hint: x^2 - y^2 = 4]$

- **Cycloids**

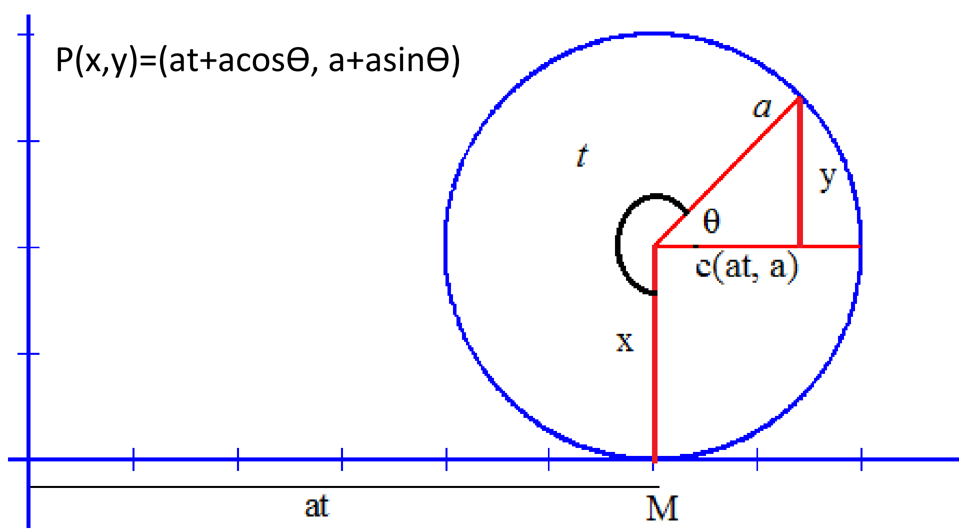
مسئلة بندول الساعة الذي يتأرجح بشكل حركة نصف دائرية ان المنحني الناتج من تذبذب حركة البندول تعتمد على سعة التأرجح, أن أوسع حركة يكون مستقرها أو مركزها هو نقطة التغيير.

في 1673 (Christian Huygens) صمم بندول الساعة, وقام بحساب التدوير للمنحني الناتج من عملية التأرجح كما في المثال التالي.

Ex: A wheel of radius a rolls along a horizontal straight line. Find parametric equations for the path traced by a point P on the wheel's circumference. The path is called a cycloid.

Sol:

لو أخذنا خط مستقيم من الأحداثي-X الى النقطة P بالنسبة للدائرة وتبدأ من النقطة P على الدائرة وصولاً الى نقطة الأصل وكأن الحلقة قد دارات الى اليمين. المقياس نستخدم زاوية التدوير t لقياس نصف الدائرة في الشكل حيث قاعدته هي at, مركزها هو النقطة (at, a) فإن احداثي المستوي للنقطة هو



$x = at + a\cos(\theta)\,, y = a + a\sin(\theta)$.

للتعبير عن الزاوية θ بالنسبة للمقياس t, فإن $t + \theta = \frac{3\pi}{2}$ كما في الشكل أعلاه

$\theta = \frac{3\pi}{2} - t$, we get that $\cos(\theta) = \cos\left(\frac{3\pi}{2} - t\right) = -\sin(t)$ also

$\sin(\theta) = \sin\left(\frac{3\pi}{2} - t\right) = -\cos(t)$. Hence $x = at - a\sin(t)$, $y = a - a\cos(t)$. Then

$x = a(t - \sin(t)), y = a(1 - \cos(t)), t \geq 0.$

وهذه هي المعادلة القياسية لحركة بندول الساعة عند أقصى نقطة $t \geq 0$.

Remark: $sin(A \pm B) = sin(A)cos(B) \pm cos(A)sin(B).$

$cos(A \pm B) = cos(A)cos(B) \mp sin(A)sin(B).$

Calculus with Parametric Curves

Tangents and Areas. A parameterized curve $x = f(t)$; $y = g(t)$ is differentiable at t, if f and g are Diff. at t.

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$ or $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$, such that $\frac{dx}{dt} \neq 0$. So $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) / \frac{dx}{dt}.$

Ex: Find the tangent to the curve.

$x = \sec(t)\; y = \tan(t)\;, -\frac{\pi}{2} < t < \frac{\pi}{2}$

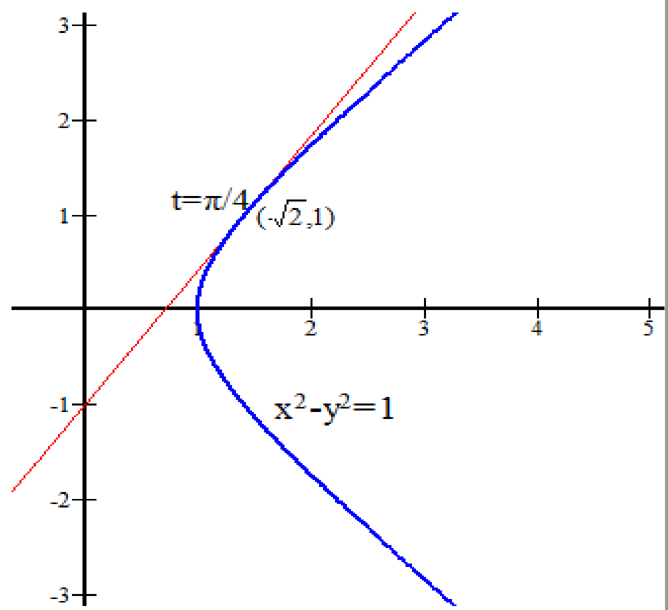
at the point $(\sqrt{2}, 1)$, where $t = \frac{\pi}{4}$.

Sol: $\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt} = \frac{\sec^2 t}{sect\; tant} = \frac{sect}{tant}$

$t = \frac{\pi}{4} \;\Rightarrow\; \frac{dy}{dx}\Big|_{\frac{\pi}{4}} = \frac{sec\left(\frac{\pi}{4}\right)}{tan\left(\frac{\pi}{4}\right)} = \frac{\sqrt{2}}{1} = \sqrt{2}\,.$

The tangent line $y - 1 = \sqrt{2}(x - \sqrt{2})$.

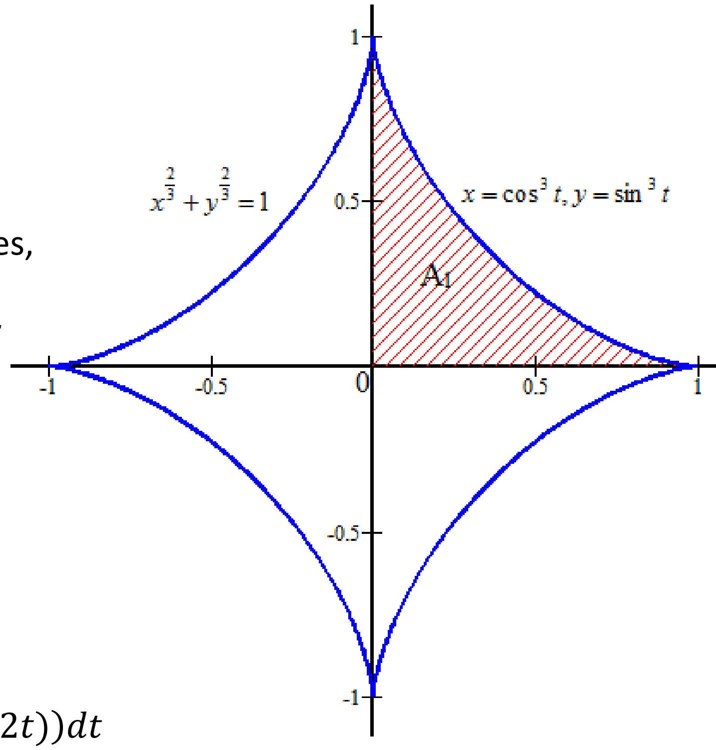
$\therefore y = \sqrt{2}x - 1.$



Ex: Find $\frac{d^2y}{dx^2}$ as a function of t, if $x = t - t^2\,, y = t - t^3$.

Sol: $y' = \frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt} = \frac{1-3t^2}{1-2t}$, but $\frac{d^2y}{dx^2} = \frac{dy'}{dt} / \frac{dx}{dt} = \frac{2-6t+6t^2}{(1-2t)^2} / (1 - 2t) = \frac{2-6t+6t^2}{(1-2t)^3}.$

Since $t = \frac{y}{x} - 1$, we have that $y'' = \frac{6\left(\left(\frac{y}{x}\right)^2 - \frac{y}{x} + \frac{7}{3}\right)}{\left(3 - \frac{2y}{x}\right)^3}$.



Ex: Find the area enclosed by the asteroid.

$x = \cos^3(t)\,, y = \sin^3(t)\,, 0 \le t \le 2\pi.$

Sol: By symmetric, the enclosed area is 4 times,

such that $0 \le t \le 2\pi$. So $A = 4A_1 = \int_0^1 y dx$

Since $y = \sin^3(t)$, $dx = 3\cos^2(t) \cdot \sin(t) dt$.

$$A = 4\int_0^{\frac{\pi}{2}} \sin^3(t)\, 3\cos^2(t) \sin(t)\, dt$$

$$= 12\int_0^{\frac{\pi}{2}} \left(\frac{1-\cos(2t)}{2}\right)^2 \left(\frac{1+\cos(2t)}{2}\right) dt$$

$$= \frac{3}{2}\int_0^{\frac{\pi}{2}} (1 - 2\cos(t) + \cos^2(2t))(1 + \cos(2t)) dt$$

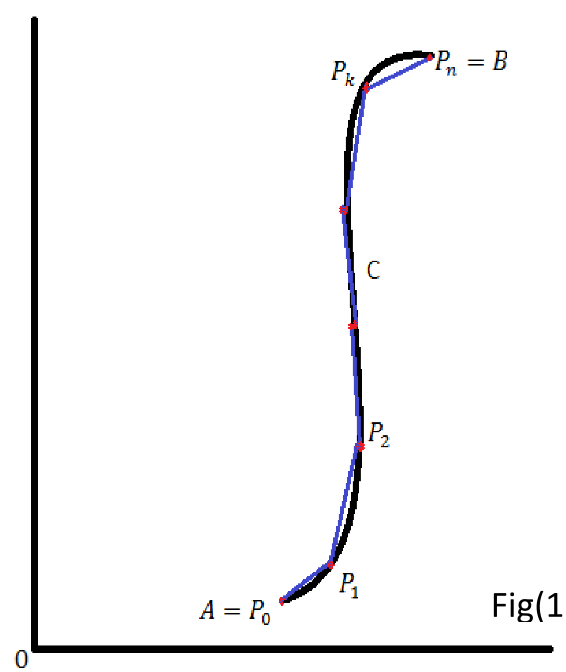
$$= \frac{3}{2}\int_0^{\frac{\pi}{2}} (1 - 2\cos(t) - \cos^2(2t) + \cos^3(2t)) dt$$

$$= \frac{3}{2}\left[\left(t - \frac{1}{2}\sin(2t)\right) - \frac{1}{2}\left(t + \frac{1}{4}\sin(2t)\right) + \frac{1}{2}\left(\sin(2t) - \frac{1}{3}\sin^3(2t)\right)\right]\Bigg|_0^{\frac{\pi}{2}} = \frac{3\pi}{8}.$$

Length of a Parametrically defined Curve

Let C be a curve given parametrically by the equation $x = f(t)\,; y = g(t), a \le t \le b$.We assume the functions f and g are continuously differentiable on the interval $[a, b]$, s.t $f'(t), g'(t)$ are exists. The smooth curve, we subdivide the path(arc) $\overline{AB}$ into n pieces at points $A = P_0, P_1, P_2, \dots, P_n = B$, the points correspond to a partition of the interval $[a, b]$ by $a = t_0 < t_1 < t_2 < \cdots < t_n = b$, where

$P_k = (f(t_k), g(t_k))$ the length $L_k = \sqrt{\Delta x_k^2 + \Delta y_k^2}$, by Fig(2) $\Delta x_k = f(t_k) - f(t_{k-1}) = f'(t_k^*)\Delta t_k$.



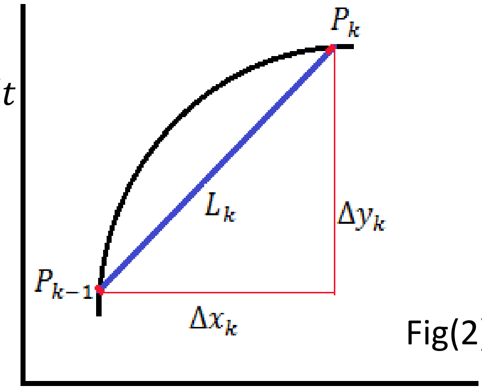
Fig(1)

$\Delta y_k = g(t_k) - g(t_{k-1}) = g'(t_k^{**})\Delta t_k. \Rightarrow \sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{\Delta x_k^2 + \Delta y_k^2}$

$= \sum_{k=1}^{n} \sqrt{f'(t_k^*)^2 + g'(t_k^{**})^2}\, \Delta t_k$ as $n \to \infty$ or $\|P\| \to 0$, we have

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} \sqrt{f'(t_k^*)^2 + g'(t_k^{**})^2}\, \Delta t_k = \int_a^b \sqrt{f'(t)^2 + g'(t)^2} dt$$

Or by Leibniz relation, we have $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$.



Fig(2)

Ex: Using the definition, find the length of the circle radius

r defined parametrically by $x = r\cos(t), y = r\sin(t), 0 \le t \le 2\pi$.

Sol: By definition, $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$, we find $\frac{dx}{dt} = -r\sin t, \frac{dy}{dt} = r\cos t$.

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = r^2(\sin^2 t + \cos^2 t) = r^2$$

$$\therefore\ L = \int_0^{2\pi} \sqrt{r^2}\, dt = rt \Big|_0^{2\pi} = 2\pi r.$$

Ex: Find the length of the asteroid $x = \cos^3(t), y = \sin^3(t), 0 \le t \le 2\pi$.

Sol: Because of the curve's symmetric w.r.t the coordinate axes, it's length is four times the length of the first-quadrant portion, we have

$$\left(\frac{dx}{dt}\right)^2 = (3\cos^2 t \cdot (-\sin t))^2 = 9\cos^4 t \cdot \sin^2 t$$

$$\left(\frac{dy}{dt}\right)^2 = (3\sin^2 t \cdot (\cos t))^2 = 9\sin^4 t \cdot \cos^2 t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{9\sin^2 t \cdot \cos^2 t(\sin^2 t + \cos^2 t)} = \sqrt{9\sin^2 t \cdot \cos^2 t} = 3\sin t\cos t$$

$$L_1 = \int_0^{\frac{\pi}{2}} 3\sin t\cos t\, dt = \frac{3}{2},\ \therefore L = 4L_1 = 4\left(\frac{3}{2}\right) = 6.$$

The Arc Length Differential

Let $x = f(t)\ ;\ y = g(t), a \le t \le b$, the arc length definition by

$S(t) = \int_a^t \sqrt{f'(z)^2 + g'(z)^2}\,dz$. Then, by the Fundamental theorem of calculus

$\frac{ds}{dt} = \sqrt{f'(t)^2 + g'(t)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$

The Diff. of arc length is $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ or $ds = \sqrt{(dx)^2 + (dy)^2}$.

Remark: The Cartesian formula, where

$x = g(y), a \le y \le b$, then $L = \int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$. Or

$y = f(x), a \le x \le b$, then $L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$.

Ex: Use the Cartesian formula to find the length of each curve. (H.W.)

1) $x = y^{\frac{3}{2}}, 0 \le y \le \frac{4}{3}$.
2) $y = \frac{3}{2}x^{\frac{2}{3}}, 0 \le x \le 1$.

Ex: Find the arc length $y = a\cosh\left(\frac{x}{a}\right)$.

Sol: $L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \Rightarrow \frac{dy}{dx} = a\sinh\left(\frac{x}{a}\right) \cdot \frac{1}{a} = \sinh\left(\frac{x}{a}\right)$

$L = \int \sqrt{1 + \left(\sinh\left(\frac{x}{a}\right)\right)^2}\, dx = \int \sqrt{\left(\cosh\left(\frac{x}{a}\right)\right)^2}\, dx = a\sinh\left(\frac{x}{a}\right) \Rightarrow \frac{L}{a} = \sinh\left(\frac{x}{a}\right)$. So $x = a\sinh^{-1}\left(\frac{L}{a}\right)$, but $\left(\frac{L}{a}\right)^2 = \sinh^2\left(\frac{x}{a}\right) = \cosh^2\left(\frac{x}{a}\right) - 1$ and $y = a\cosh\left(\frac{x}{a}\right)$.

$\therefore \left(\frac{L}{a}\right)^2 = \left(\frac{y}{a}\right)^2 - 1$. Then $y = \sqrt{L^2 + a^2}$.

Ex: Show that $y = a\cosh\left(\frac{x}{a}\right)$ satisfies, the Diff. Eq. $\frac{d^2y}{dx^2} = \frac{w}{H}\sqrt{1+\left(\frac{dy}{dx}\right)^2}$, provided $a = \frac{H}{w}$.

Sol: $y = a\cosh\left(\frac{x}{a}\right)$, so $y' = a\sinh\left(\frac{x}{a}\right)\cdot\frac{1}{a} = \sinh\left(\frac{x}{a}\right)$ also $y'' = \frac{1}{a}\cosh\left(\frac{x}{a}\right)$, by substitution the equation, we have

$$\frac{d^2y}{dx^2} = \frac{1}{a}\cosh\left(\frac{x}{a}\right) = \frac{w}{H}\sqrt{1+\left(\sinh\left(\frac{x}{a}\right)\right)^2} = \frac{w}{H}\sqrt{\left(\cosh\left(\frac{x}{a}\right)\right)^2}$$

$\Rightarrow \frac{1}{a}\cosh\left(\frac{x}{a}\right) = \frac{1}{a}\cosh\left(\frac{x}{a}\right)$. Then $y = a\cosh\left(\frac{x}{a}\right)$ satisfies.

Ex: For the Fig(*). Show below, show that

$$\frac{d^2y}{dx^2} = \frac{w}{H}\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

where H=Horizontal tension force and w=weight for unit length.

Sol: $H = T\cos\theta$ ...(1); $ws = T\sin\theta$ ...(2) حسب تحليل المركبات الجيبية

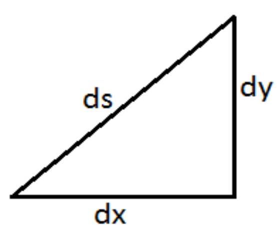
Dividing (2) by (1), we have $\frac{T\sin\theta}{T\cos\theta} = \tan\theta = \frac{ws}{H}$,

the slop $\therefore\ \frac{dy}{dx} = \tan\theta = \frac{ws}{H} \Rightarrow$

$\frac{d^2y}{dx^2} = \frac{w}{H}\frac{ds}{dx}$ ...(3), so $\frac{ds}{dx} = \frac{H}{w}\frac{d^2y}{dx^2}$, since

$(ds)^2 = (dx)^2 + (dy)^2$,

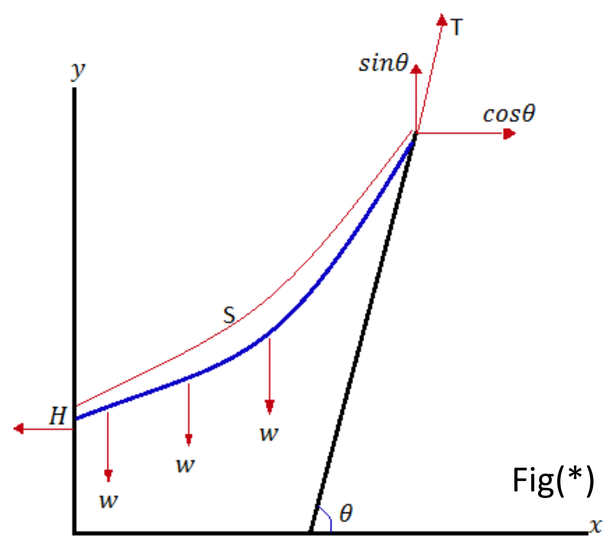
therefore



$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$, thus

$\left(\frac{ds}{dx}\right) = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$, we put in (3). Then

$\frac{d^2y}{dx^2} = \frac{w}{H}\sqrt{1+\left(\frac{dy}{dx}\right)^2}$. وهذه تمثل معادلة تفاضلية لإيجاد قوة الشّد



Fig(*)

Area of Surfaces of Revolution

If a smooth curve $x = f(t)$; $y = g(t), a \leq t \leq b$, is traversed exactly once as t increases from a to b, then the area of the Surfaces generated by revolving the cure about the coordinate axes are as follows.

1) Revolution about the X-axis ($y \geq 0$)

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

2) Revolution about the Y-axis ($x \geq 0$)

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Ex: The standard parameterization of the circle of radius 1 centered at the point (0, 1) in the XY-plane is $x = cost\, , y = 1 + sint\, , 0 \leq t \leq 2\pi$. Use this parameterization to find the area of the surface swept out by revolution the circle about the X-axis.

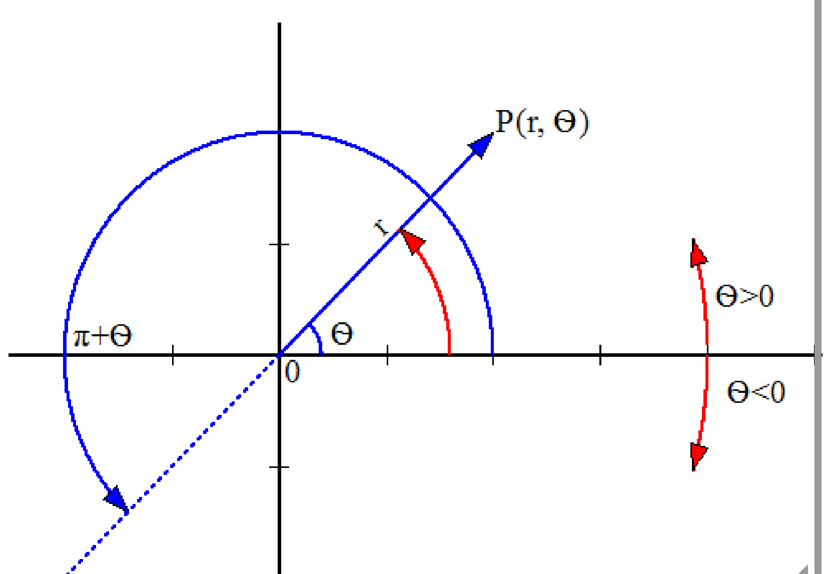
Sol: We evaluate the formula

$$S = \int_0^{2\pi} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} 2\pi(1 + sint\,) \sqrt{(-sint)^2 + (cost)^2} dt$$

$$= 2\pi \int_0^{2\pi} (1 + sint\,)\, dt = 2\pi(t - cost)\Big|_0^{2\pi} = 4\pi^2$$

Polar Coordinates

To define polar coordinates, we first fix an origin 0 (called the pole) and an initial ray from 0 Fig(4). Then each point P can be located by assigning to it a Polar-Coordinate pair (r, θ) in which r gives the directed distance from 0 to P and θ gives the directed angle from the initial ray to ray $0P$.



Ex: Find all the polar coordinates of the point $P\left(2,\frac{\pi}{6}\right)$.

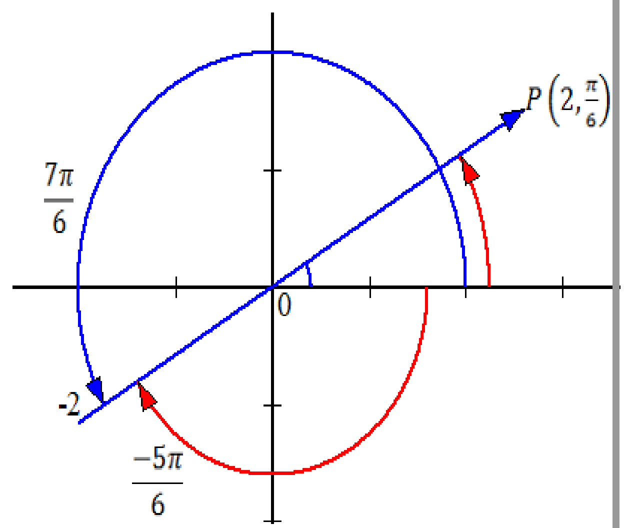
Sol: $r=2$, the complete list of angles is $\frac{\pi}{6},\frac{\pi}{6}\pm 2\pi,\frac{\pi}{6}\pm 4\pi,\ldots$

For $r=-2$, the angles are

$-\frac{5\pi}{6},-\frac{5\pi}{6}\pm 2\pi,-\frac{5\pi}{6}\pm 4\pi,\ldots$

The corresponding coordinate pair of pare $\left(2,\frac{\pi}{6}\pm 2n\pi\right)$ and $\left(-2,-\frac{5\pi}{6}\pm 2n\pi\right)$,

$n=0,\pm1,\pm2,\ldots,$ when $n=0$, the formula give $\left(2,\frac{\pi}{6}\right)$ and $\left(-2,-\frac{5\pi}{6}\right)$, when $n=1$, they give $\left(2,\frac{13\pi}{6}\right)$ and $\left(-2,\frac{7\pi}{6}\right)$, and so on.



Remark: في الاحداثيات القطبية يمكن ان تعين بأكثر من طريقة واحدة فمثلاً

1) النقطة P يمكن أن تعين بـ(r,θ) أو $(-r,\pi+\theta)$, أي يمكن قراءة النقطة الواحدة بعدد لانهائي من الأعداد وبصورة عامة يمكن كتابة الصيغة التالية لقراءة النقطة في الاحداثيات القطبية.

$$(r,\theta+2n\pi),(-r,(\theta\pm\pi)+2n\pi),n=0,\pm1,\pm2,\ldots$$

2) للمنحني الواحد أكثر من معادلة واحدة, أي توجد عدة معادلات متكافئة, وللحصول على المعادلات المتكافئة نستعمل التعويض

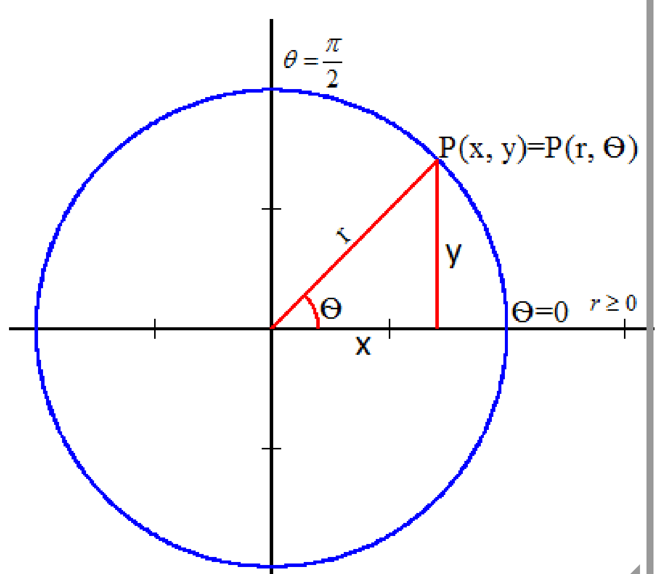
$$r\rightarrow -r\,;\,\theta\rightarrow \pi+\theta$$

3) لتحويل الاحداثيات الكارتيزية (x,y) الى القطبية (r,θ) أو بالعكس, نستخدم المعادلة التالية:

Equations relating polar and Cartesian coordinates:

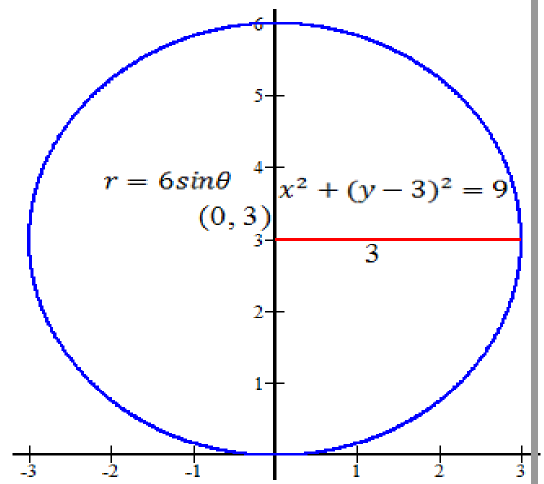
$$x=rcos\theta, y=rsin\theta$$

$$r^2=x^2+y^2, tan\theta=\frac{y}{x}$$



Ex: Find a polar equation for the circle

$x^2 + (y-3)^2 = 9.$



Sol: $x^2 + (y-3)^2 = 9$

$x^2 + y^2 - 6y + 9 = 9$

$x^2 + y^2 = 6y \Rightarrow r^2 = 6rsin\theta \Rightarrow r = 6sin\theta.$

Ex: Replace the following polar equation by equivalent Cartesian equation and identify their graphs. (H.W.)

1) $rcos\theta = -4$ 2) $r^2 = 4rcos\theta$ 3) $= \frac{4}{2cos\theta - sin\theta}$.

Ex: Show that $r = cos\theta + 1$ and $r = cos\theta - 1$, represent the same curve.

Sol: $r = cos\theta + 1 \Rightarrow -r = \cos(\theta + \pi) + 1$

$\Rightarrow -r = [\cos\theta cos\pi - sin\theta sin\pi] + 1$

$\Rightarrow -r = -\cos(\theta) + 1, \Rightarrow r = \cos(\theta) - 1.$

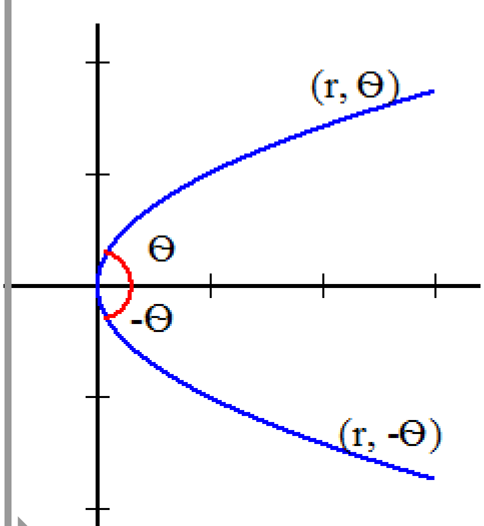
Ex: Replace $P(r,\theta) \to P(x,y)$ and graph. $rcos\left(\theta - \frac{\pi}{3}\right) = 3$. (H.W.)

Graphing in Polar Coordinates

Symmetry tests for polar graphs

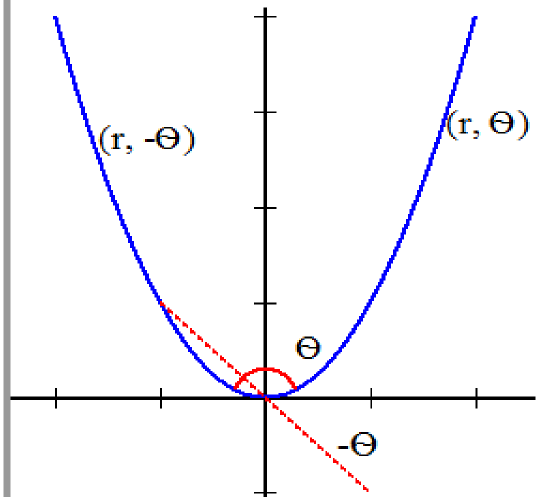
1) Symmetry about the X-axis: if the point(r,θ) lies on the graph then the point $(r,-\theta)$ or $(-r,\pi-\theta)$ lies on the graph.

- أي منحني يحتوي على $\cos\theta$ فقط يكون متناظر حول محور x كونها زوجية $\cos(-\theta) = \cos\theta$



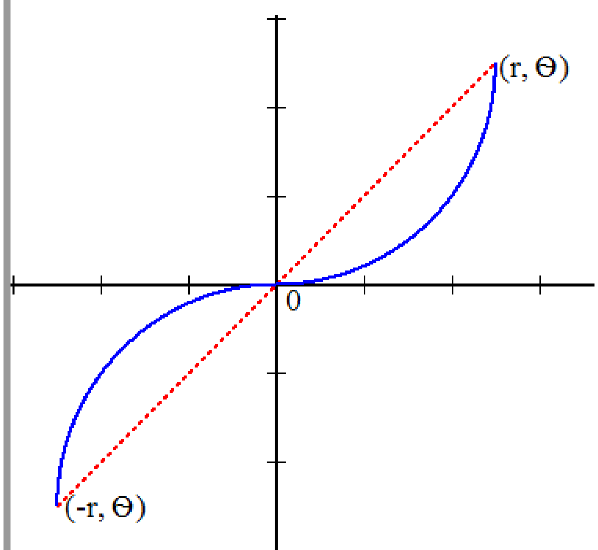
2) Symmetry about the Y-axis: If the point(r,θ) lies on the graph then the point $(r,\pi-\theta)$ or $(-r,-\theta)$ lies on the graph.

- أي منحني يحتوي على $sin\,\theta$ فقط يكون متناظر حول محور y كونها فردية $sin(-\theta)=-sin\,\theta$



3) Symmetry about the origin: If the point(r,θ) lies on the graph then the point $(-r,\theta)$ or $(r,\pi+\theta)$ lies on the graph.

- أي منحني يحتوي على r^2 فقط, يكون متناظر حول نقطة الأصل.



<u>**Remark**</u>:عند رسم المنحني في المحاور القطبية فإن

1) أذا كان المنحني متناظر حول المحور X فتؤخذ حدود θ من $0 \leftarrow \pi$

2) أذا كان المنحني متناظر حول المحور Y فتؤخذ حدود θ من $-\frac{\pi}{2} \leftarrow \frac{\pi}{2}$

3) أذا كان المنحني متناظر حول كلمن المحاور X,Y (نقطة الاصل)فتؤخذ حدود θ من $0 \leftarrow \frac{\pi}{2}$.

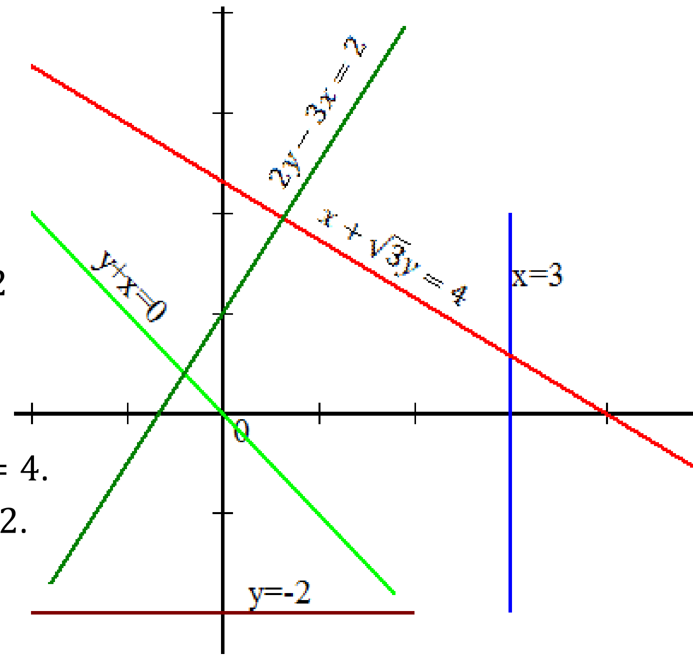
- **(1) Lines in Polar Coordinates**

$$ax + by = c \longrightarrow r(acos\theta + bsin\theta) = c \qquad a, b, c \in R$$

Ex: Sketch the following in polar coordinates

1) $rcos\theta = 2$ 2) $r = 3sec\theta$ 3) $r = 2\sec\theta$ 4) $r = -2csc\theta$
5) $\theta = \frac{3\pi}{4}$ 6) $r = \frac{2}{2sin\theta - 3cos\theta}$.

Sol:



1) $rcos\theta = 2 \longrightarrow x = 2$.
2) $r = 3sec\theta \Rightarrow rcos\theta = 3 \longrightarrow x = 3$.
3) $r = 2\sec\theta \Rightarrow rcos\left(\theta - \frac{\pi}{3}\right) = 2$

$$\Rightarrow r\left(cos\theta \cos\left(\frac{\pi}{3}\right) + sin\theta \sin\left(\frac{\pi}{3}\right)\right) = 2$$

$$\Rightarrow r\left(\frac{1}{2}cos\theta + \frac{\sqrt{3}}{2}sin\theta\right) = 2$$

$$\Rightarrow rcos\theta + \sqrt{3}sin\theta = 4 \longrightarrow x + \sqrt{3}y = 4.$$

4) $r = -2csc\theta \Rightarrow rsin\theta = -2 \longrightarrow y = -2$.
5) $\theta = \frac{3\pi}{4}$, since $\theta = \tan^{-1}\left(\frac{y}{x}\right)$,

so $\tan^{-1}\left(\frac{y}{x}\right) = \frac{3\pi}{4} \Rightarrow \frac{y}{x} = \tan\left(\frac{3\pi}{4}\right) \Rightarrow \frac{y}{x} = -1 \Rightarrow y + x = 0$.

6) $r = \frac{2}{2sin\theta - 3cos\theta} \Rightarrow r(2sin\theta - 3cos\theta) = 2 \longrightarrow 2y - 3x = 2$.

Remark:

$rcos\theta = a \longrightarrow$ line // y-axis.

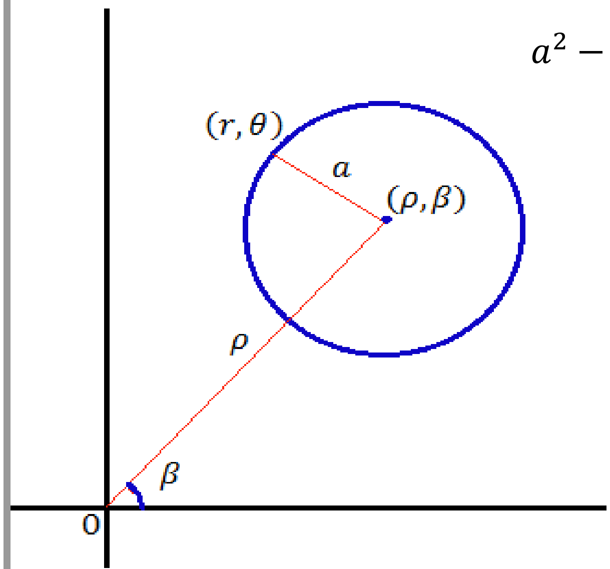
$rsin\theta = a \longrightarrow$ line // x-axis.

$\theta = \theta_0 \longrightarrow$ line through origin with angle θ_0.

- (2) **Circles in Polar Coordinates**

$(x-h)^2+(y-k)^2=a^2 \quad \longrightarrow \quad r^2=a^2-\rho^2+2r\rho\cos(\theta-\beta)$

$a^2-\rho^2=r^2-2r\rho\cos(\theta-\beta)$; where



ρ: بعد المركز عن نقطة الاصل

a : نصف قطر الدائرة

(r,θ): أحداثيات نقطة على محيط الدائرة

β: زاوية التدوير

- Spatial case:

1) When $\rho=a, r^2=2r\rho\cos(\theta-\beta)$ or $r=2\rho\cos(\theta-\beta)$, so if $\beta=0, r=2\rho\cos\theta$also if $\beta=\frac{\pi}{2}, r=2\rho\sin\theta$.
2) If $\rho=0$, we have $r=a$.

Ex: Sketch the following in polar coordinates: (H.W.)

1) $r=4\cos\theta$ 2) $r=-4\cos\theta$ 3) $r=5\sin\theta$ 5) $r^2=9$.

Lemma: Let $a,b\in R$ and α,θ are angles, then

$a\sin\theta+b\cos\theta=\sqrt{a^2+b^2}\cos(\theta-\alpha)$, such that $\alpha=\tan^{-1}\left(\frac{a}{b}\right)$.

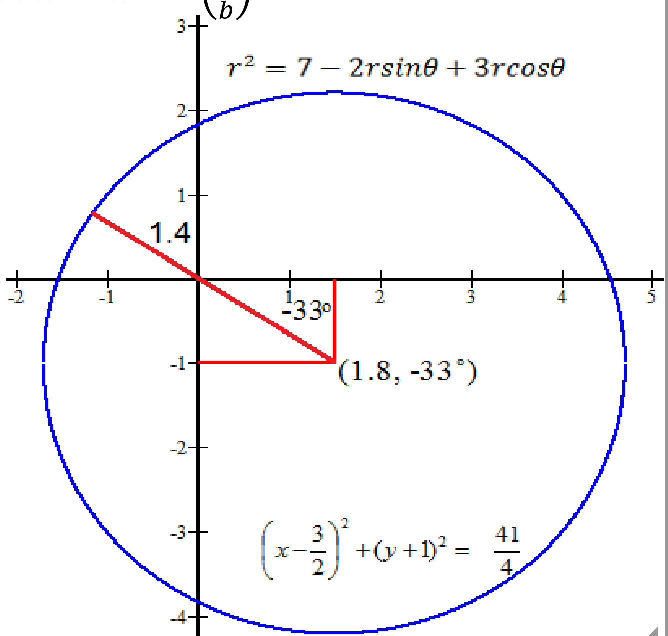
Ex: Sketch $r^2=7-2r\sin\theta+3r\cos\theta$.

Sol: $r^2=7+(-2r\sin\theta)+3r\cos\theta$

$=7+\sqrt{(-2)^2+3^2}r\cos(\theta-\alpha)$;

$\alpha=\tan^{-1}\left(-\frac{2}{3}\right)=-33°$,

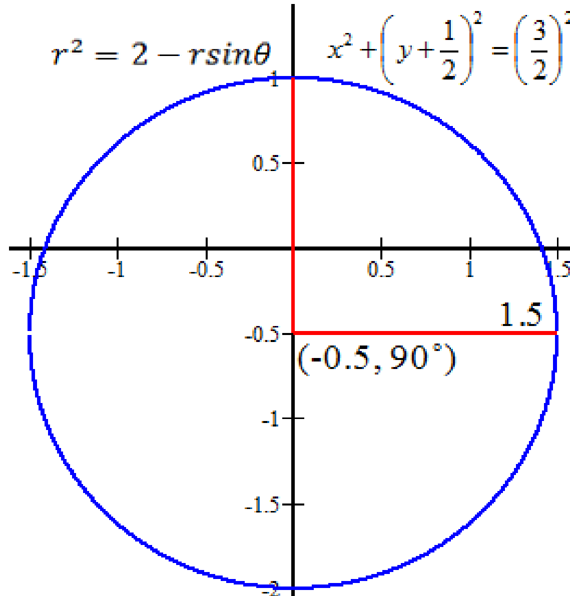
$r^2=7+\sqrt{13}r\cos(\theta+33°)$.



By Circle Eq. $a^2-\rho^2=r^2-2r\rho\cos(\theta-\beta)$; we have $2\rho=\sqrt{13}\Rightarrow\rho=1.8$, also $a^2-\rho^2=7\ \Rightarrow a^2=\frac{41}{4}\ \Rightarrow a=3.2$ radius, $\beta=-33°$, so center $(\rho,\beta)=(1.8,-33°)$.

Ex: Sketch

1) $r^2=2-rsin\theta$ 2) $r=2sin\theta+5cos\theta$



Sol: 1) $r^2=2-rsin\theta=2-rcos\left(\theta-\frac{\pi}{2}\right)$

$\Rightarrow 2=r^2+rcos\left(\theta-\frac{\pi}{2}\right)$, so

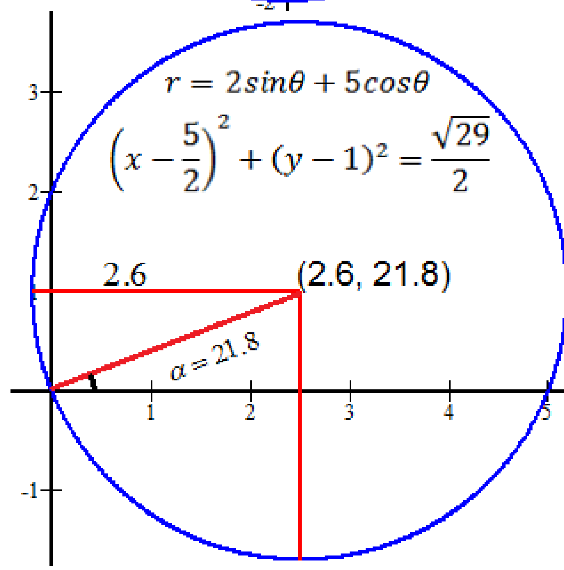
$-2\rho=1\ \Rightarrow\ \rho=\left|-\frac{1}{2}\right|=\frac{1}{2}\ ,\beta=\frac{\pi}{2}.$

$\therefore$ center $(\rho,\beta)=\left(-\frac{1}{2},\frac{\pi}{2}\right)$, $a^2-\rho^2=2\Rightarrow a=\frac{3}{2}$.

2) $r=2sin\theta+5cos\theta$, by lemma, we have that

$$r=\sqrt{4+25}\cos(\theta-\alpha)\ ;\ \alpha=\tan^{-1}\left(\frac{2}{5}\right)=21.8°$$

$$\Rightarrow r=\sqrt{29}\cos(\theta-21.8°).$$



First graphing the curve $r=\sqrt{29}\cos\theta$, and shifting angle by 21.8°. Hence $\rho=a$

$\Rightarrow 2a=\sqrt{29}\ \Rightarrow a=2.6\ ;C(2.6,21.8°).$

Remark: رسم المنحني$r=f(\theta-\alpha)$, ناتج عن تدوير المنحني $r=f(\theta)$ بزاوية α مقاسة بالاتجاه الموجب للزاوية(عكس عقارب الساعة).

- **(3) Lima cons and Cardioids**

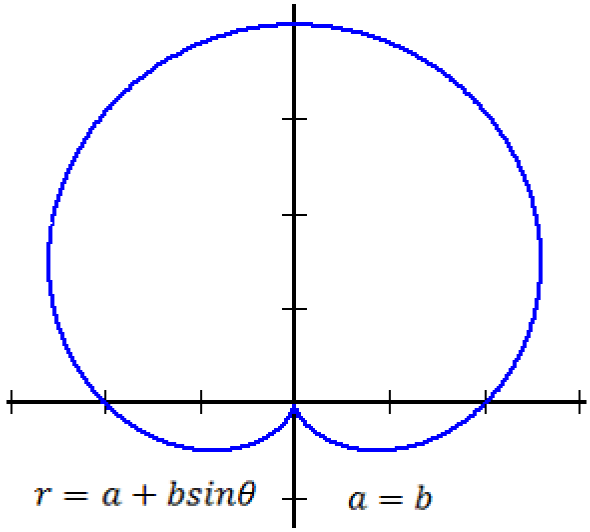
Equations of the form

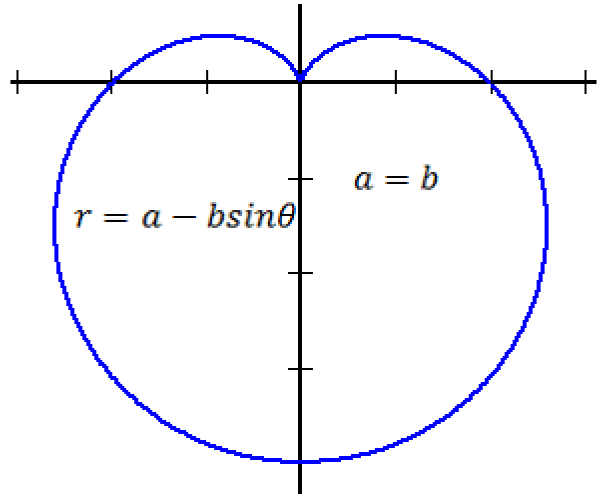
$r=a\pm bsin\theta\ ...(1)$

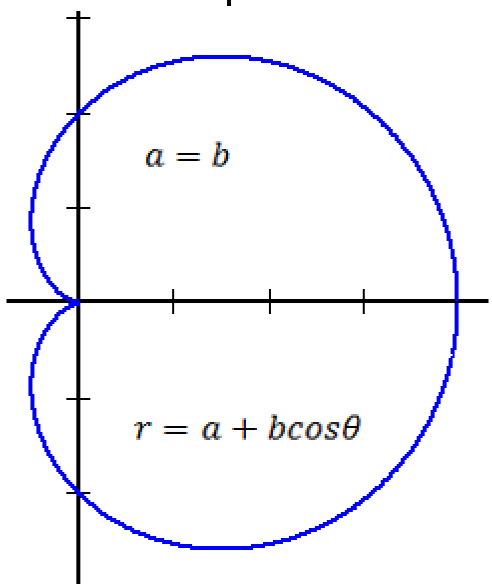
$r=a\pm bcos\theta\ ...(2)$

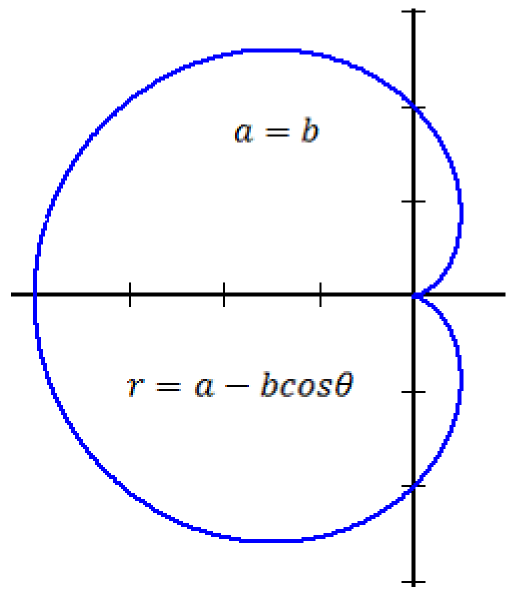
توجد أربعة إحتمالات لشكل (Lima cons) وذلك نسبتاً للمقدار $\frac{a}{b}$. $\left[\frac{a}{b}<1;\frac{a}{b}=1;1<\frac{a}{b}<2;\frac{a}{b}\geq 2\right]$

Lima cons are called “Cardioid”, when $a = b$.







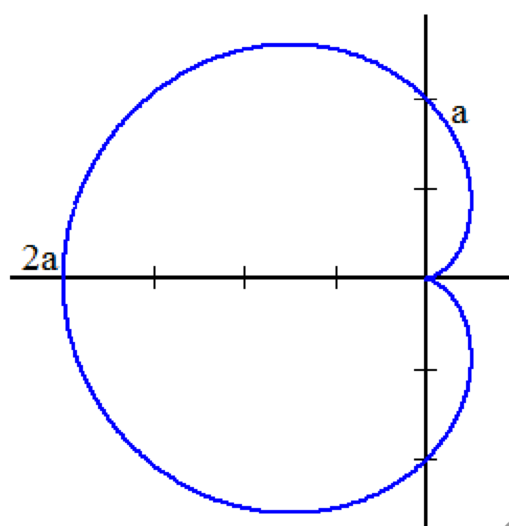


Ex: Sketch the following graphs:

1) $r = a(1 - cos\theta)$ 2) $r = a(1 + cos\theta)$

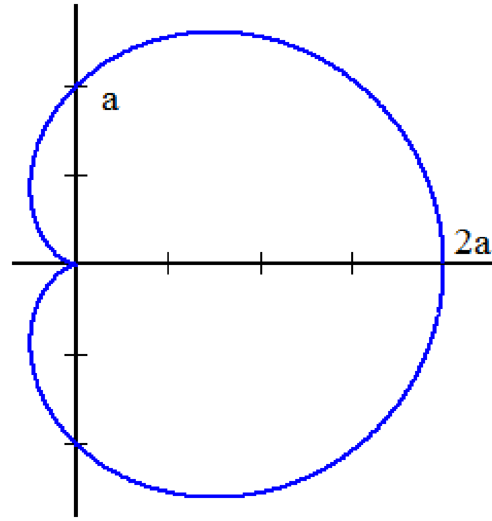
3) $r = a(1 + sin\theta)$ 4) $r = a(1 - sin\theta)$

Sol: 1) $r = a(1 - cos\theta)$, $0 \le \theta \le \pi$, since the curve symmetric about X-axis



θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	0	$\frac{a}{2}$	a	$\frac{3a}{2}$	$2a$

2) $r = a(1 + cos\theta)$, $0 \leq \theta \leq \pi$, since the curve symmetric about X-axis



θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	$2a$	$\frac{3a}{2}$	a	$\frac{a}{2}$	0

3) & 4) (H.W.)

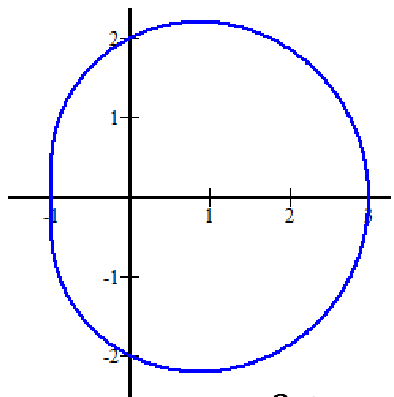
Ex: Sketch the following:

1) $r = 1 + cos\theta$ 2) $r = 2 + cos\theta$ 3) $r = 1 + 2cos\theta$

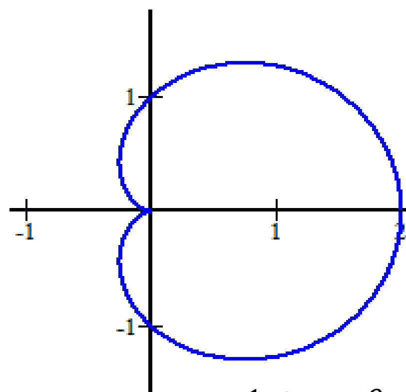
4) $r = 2 - cos\theta$ 5) $r = 1 - 2cos\theta$

Sol: 1) $= 1 + cos\theta$, symmetric about X-axis and $a = b$

θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	2	$\frac{3}{2}$	1	$\frac{1}{2}$	0



$r = 2 + cos\theta$



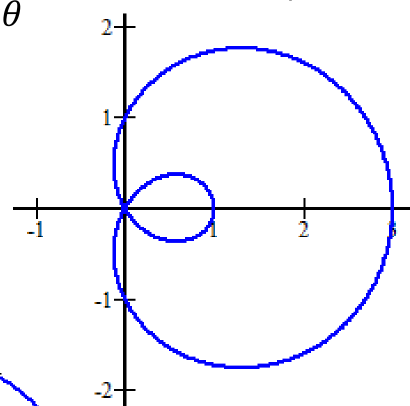
$r = 1 + cos\theta$

2) $r = 2 + cos\theta$, then $\frac{a}{b} = 2$

θ	0	$\frac{\pi}{2}$	π
r	3	2	1

3) $r = 1 + 2cos\theta$, Symmetric about X-axis $\frac{a}{b} = \frac{1}{2}$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	3	2.7	2	1	0	-1

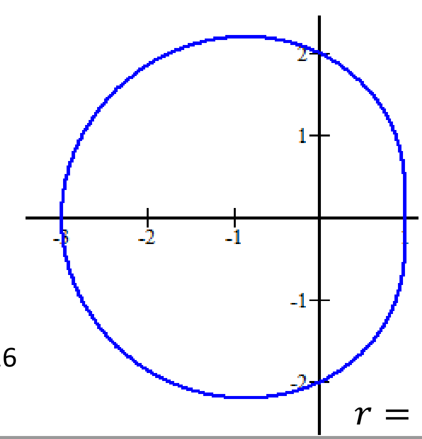


$r = 1 + 2cos\theta$

4) $r = 2 - cos\theta$, then $\frac{a}{b} = 2$, Symmetric about X-axis

θ	0	$\frac{\pi}{2}$	π
r	1	2	3

5) $r = 1 - 2cos\theta$, $\frac{a}{b} = \frac{1}{2}$. (H.W.)



$r = 2 - cos\theta$

- **(4) Lemniscates**

If $a > 0$, then the equations of the form

$r^2 = \pm a^2 \cos(2\theta) \ldots (1)$
$r^2 = \pm a^2 \sin(2\theta) \ldots (2)$

are called "Lemniscate". Leminscate always symmetric about {X-axis, Y-axis, Origin}.

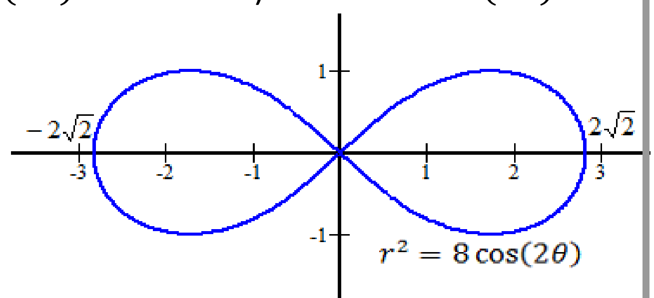
Remark:

- $-cos\theta = \cos(\theta - \pi)$ also $-\cos(2\theta) = \cos 2\left(\theta - \frac{\pi}{2}\right)$.
- $sin\theta = \cos\left(\theta - \frac{\pi}{2}\right)$ also $\sin(2\theta) = \cos 2\left(\theta - \frac{\pi}{4}\right)$.

Ex: Sketch the following:

1) $r^2 = 8\cos(2\theta)$ 2) $r^2 = -9\cos(2\theta)$ 3) $r^2 = -16\sin(2\theta)$

Sol: Let α the angle od around, then



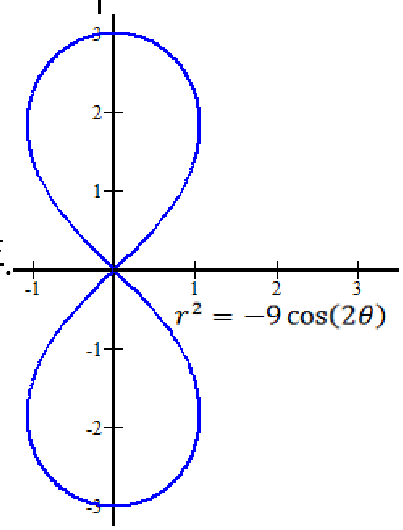
1) $r^2 = 8\cos(2\theta)$, then $a^2 = 8$,

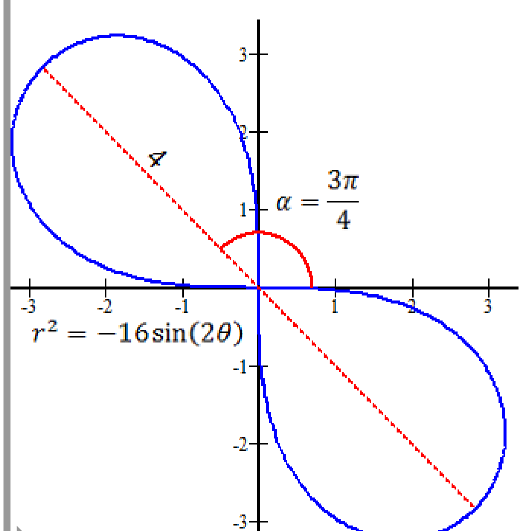
so $a = 2\sqrt{2}$. $\alpha = 0$.

2) $r^2 = -9\cos(2\theta)$, then $-a^2 = -9$, so $a = 3$.

Or $r^2 = 9\cos 2\left(\theta - \frac{\pi}{2}\right)$. $\alpha = \frac{\pi}{2}$.

3) $r^2 = -16\sin(2\theta)$, then $-a^2 = -16$, so $a = 4$. Or
$r^2 = -16cos2\left(\theta - \frac{\pi}{4}\right)$. Therefore $r^2 = 16cos2\left(\theta - \frac{3\pi}{4}\right)$. $\alpha = \frac{3\pi}{4}$.

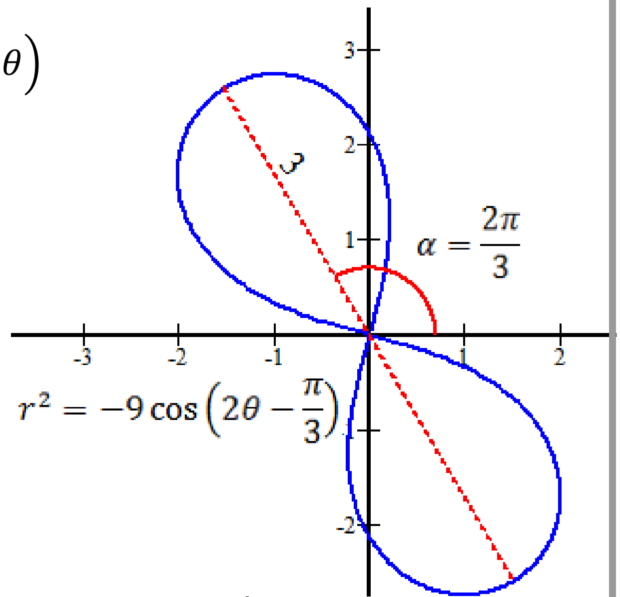




Ex: Sketch the following:

1) $r^2 = -9\cos\left(2\theta - \frac{\pi}{3}\right)$ 2) $r^2 = 16\cos\left(\frac{\pi}{4} - 2\theta\right)$

3) $r^2 = 8\cos^2\theta - 4$ 4) $r^2 = 9 - 18\sin^2\theta$



Sol: 1) $r^2 = -9\cos\left(2\theta - \frac{\pi}{3}\right)$

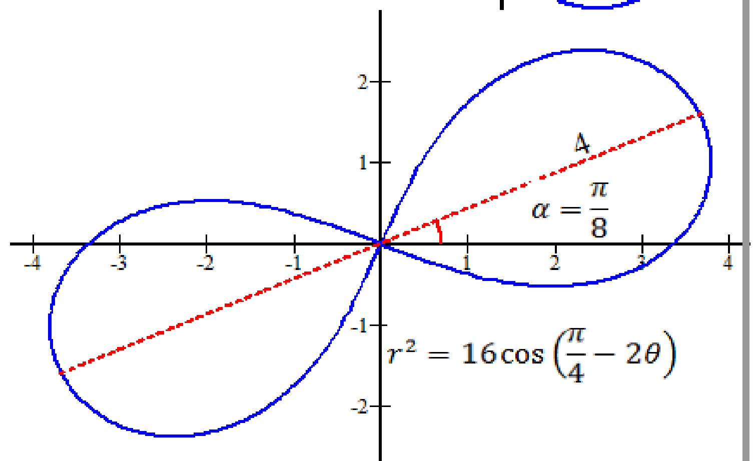
$$\Rightarrow\ r^2 = -9\cos 2\left(\theta - \frac{\pi}{6}\right)$$

$$\Rightarrow\ r^2 = 9\cos 2\left(\theta - \frac{2\pi}{3}\right)$$

So $\alpha = \frac{2\pi}{3}$.

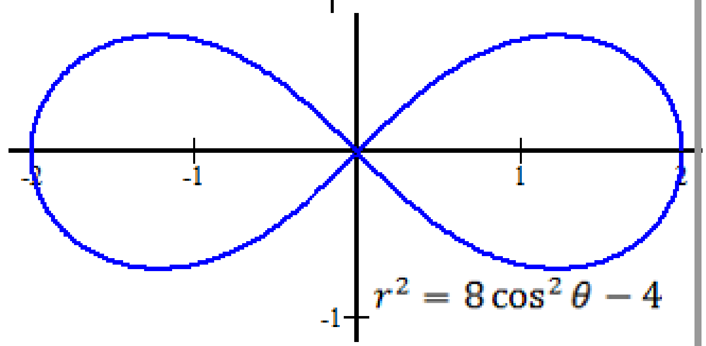
2) $r^2 = 16\cos\left(\frac{\pi}{4} - 2\theta\right)$

$$\Rightarrow\ r^2 = 16\cos 2\left(\theta - \frac{\pi}{8}\right).\ \alpha = \frac{\pi}{8}.$$



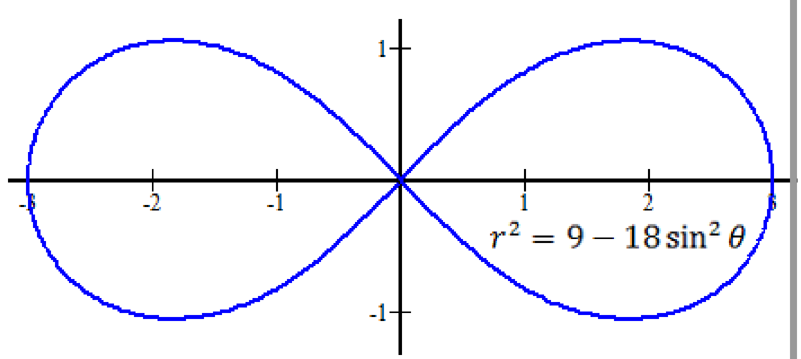
3) $r^2 = 8\cos^2\theta - 4$

$$\Rightarrow\ r^2 = 4(2\cos^2\theta - 1) = 4\cos(2\theta).$$



4) $r^2 = 9 - 18\sin^2\theta$

$$\Rightarrow\ r^2 = 9(1 - 2\sin^2\theta) = 9\cos(2\theta).$$



Ex: Sketch the following: (H.W.)

1) $r^2 = \pm a^2 sin\theta$ 2) $r^2 = \pm a^2 \cos\theta$ 3) $r^2 = 9\cos\left(\theta + \frac{\pi}{6}\right)$.

- **(5) Rose Curve**

Equations of the form

$r = asin(n\theta)\ n > 1\ , n \in Z^+$

$r = acos(n\theta)\ n > 1\ , n \in Z^+$

Represent flower-shaped curves called Roses

Remark:

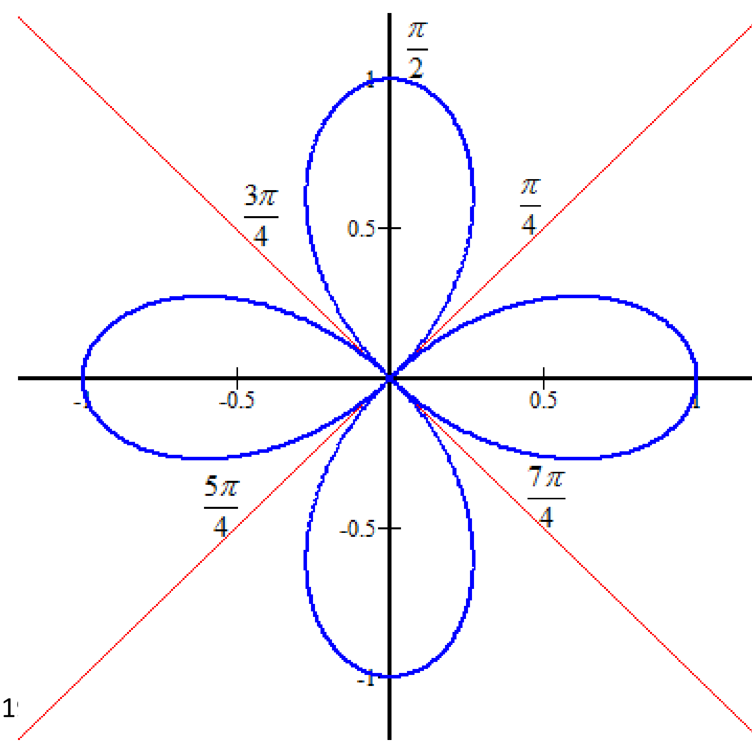
- يكون رسم وردة ذات (n) ورقة اذا كان n عدداً فردياً.
- يكون رسم وردة ذات (2n) ورقة اذا كان (n) عدداً زوجياً.

Ex: Sketch $r = cos2\theta$

Sol: when $r = 0 \Rightarrow cos2\theta = 0 \Rightarrow 2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \ldots$

So $r = 1 \Rightarrow cos2\theta = 1 \Rightarrow 2\theta = 0, \frac{4\pi}{2}, \frac{8\pi}{2}, \frac{12\pi}{2}, \ldots \Rightarrow \theta = 0, \pi, 2\pi, 3\pi, \ldots$

also $r = -1 \Rightarrow cos2\theta = -1 \Rightarrow 2\theta = \frac{2\pi}{2}, \frac{6\pi}{2}, \frac{10\pi}{2}, \ldots \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

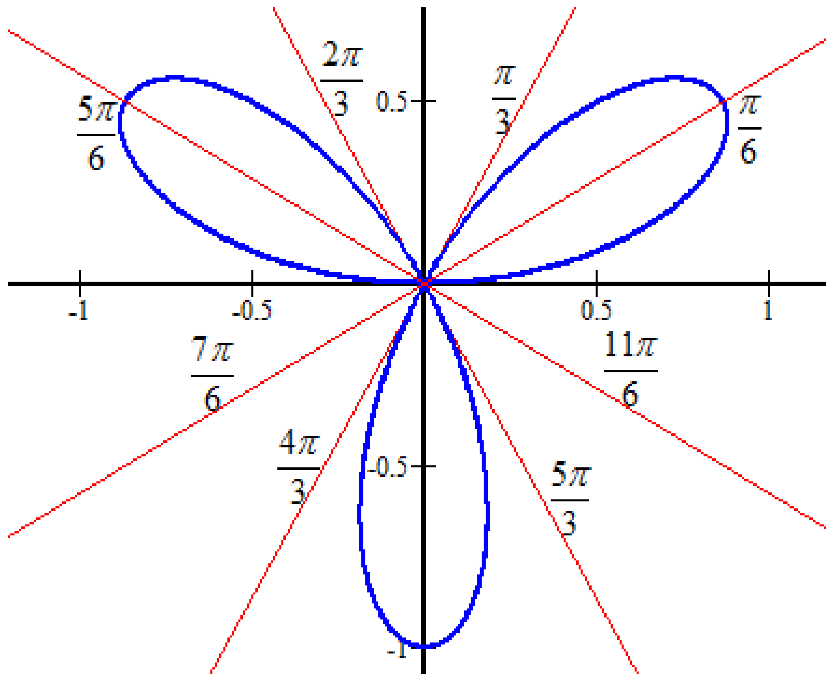


Ex: Sketch $r = sin3\theta$ 3-leafs

Sol: when $r = 0 \Rightarrow sin3\theta = 0 \Rightarrow 3\theta = 0, \frac{2\pi}{2}, \frac{4\pi}{2}, \frac{6\pi}{2}, \ldots \Rightarrow \theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \ldots$

So $r = 1 \Rightarrow sin3\theta = 1 \Rightarrow 3\theta = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots \Rightarrow \theta = \frac{3\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \ldots$

also $r = -1 \Rightarrow sin3\theta = -1 \Rightarrow 3\theta = \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}, \ldots \Rightarrow \theta = \frac{3\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \ldots$



- **(6) Spirals**

Equations of the form

$r = a\theta\ ,\ \theta \geq 0$ anticlockwise

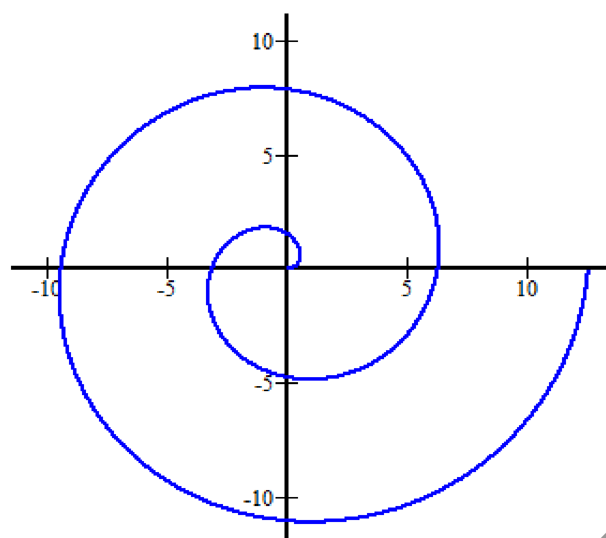
$r = a\theta\ ,\ \theta < 0$ clockwise

- عند رسم اي منحني قطبي لايحتوي على اي علاقة مثلثية, يجب تعويض قيم θ الموجبة والسالبة, ولانكتفي بدورة واحدة. وهذا الرسم عادةً يكون شكل حلزون (وتحسب بـradian).

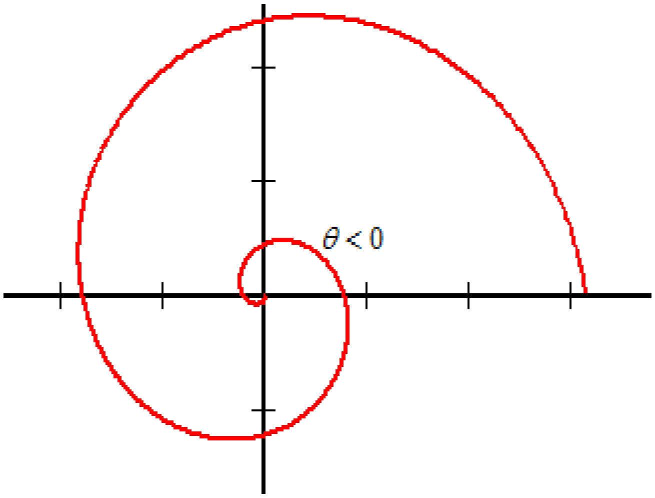
Ex: Sketch $r = \theta\ , \theta \geq 0$, symmetric about Y-axis

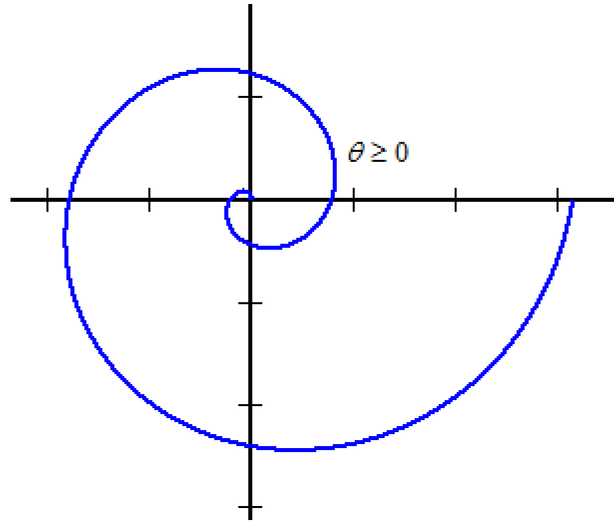
Intersection points of X-axis $r = \theta = 0, \pi, 2\pi, 3\pi, \ldots$

Intersection points of Y-axis $r = \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots$

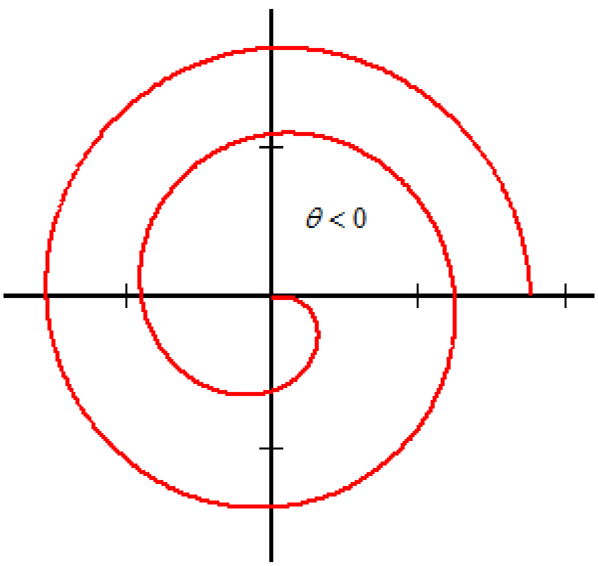


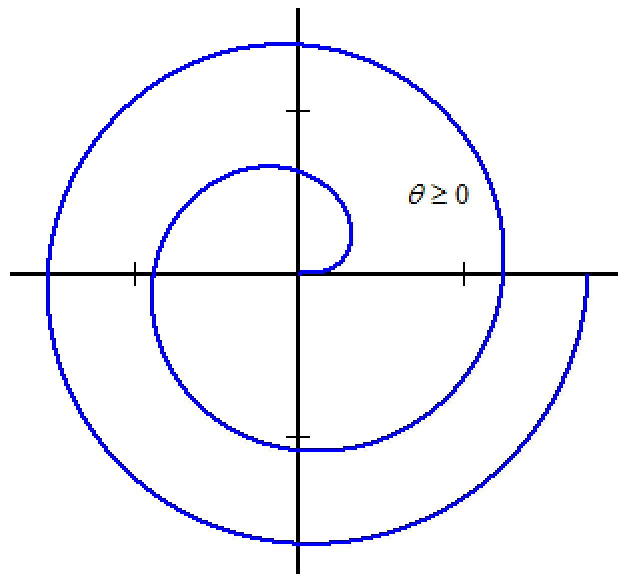
Ex: Sketch $r = \theta^2$





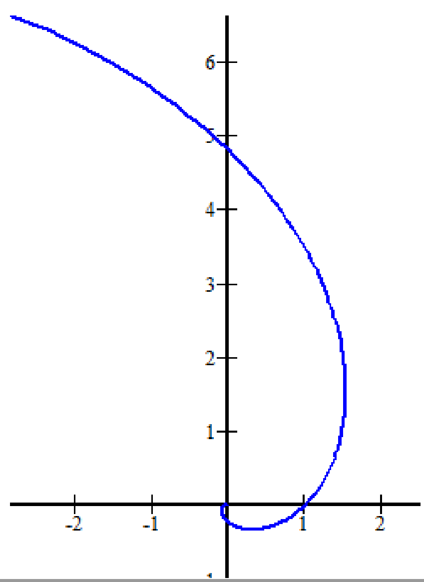
Ex: Sketch $r = \pm\sqrt{\theta}$





Ex: Sketch $r = e^{\theta}$

Sol: the $r = e^{\theta}$ has no symmetric



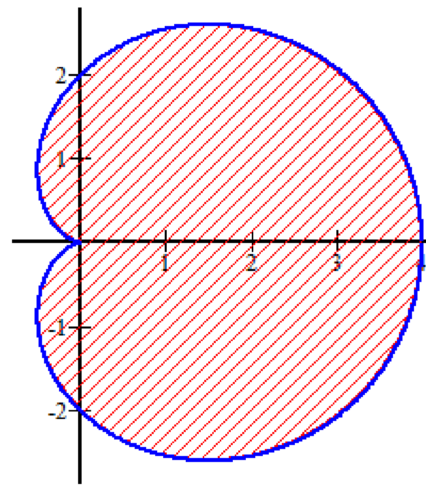
 # Areas and Length in Polar Coordinates

Area of the Fan-shaped region between the origin and the curve $r = f(\theta), \alpha \le \theta \le \beta$.

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$

This is the integral of the area differential $dA = \frac{1}{2} r^2 d\theta = \frac{1}{2}\big(f(\theta)\big)^2 d\theta$.

Ex: Find the area of the region in the plane enclosed by the cardioid $r = 2 + 2cos\theta$.



Sol: $A = \int_0^{2\pi} \frac{1}{2} r^2 \, d\theta = \int_0^{2\pi} \frac{1}{2} 4(1 + cos\theta)^2 \, d\theta$

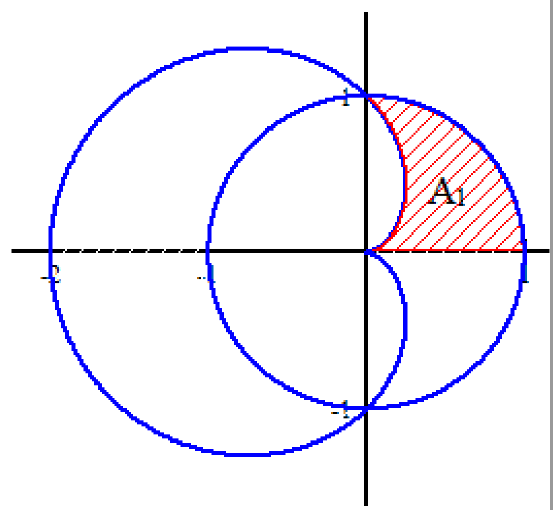
$= \int_0^{2\pi} (2 + 4cos\theta + cos2\theta) \, d\theta = 6\pi.$

If $r = a(1 + cos\theta)$ $A = ?$

- Area of the region $0 \le r_1(\theta) \le r_2(\theta)$, $\alpha \le \theta \le \beta$.

$$A = \int_{\alpha}^{\beta} \frac{1}{2} (r_2(\theta))^2 \, d\theta - \int_{\alpha}^{\beta} \frac{1}{2} (r_1(\theta))^2 \, d\theta = \int_{\alpha}^{\beta} \frac{1}{2} ({r_2}^2 - {r_1}^2) \, d\theta$$

Ex: Find the area of the region that lies inside the Circle $r = 1$ and outside the Cardioid $r = 1 - cos\theta$.



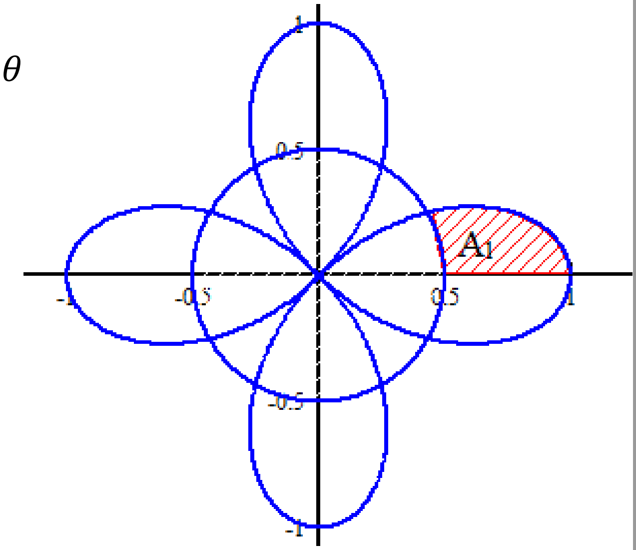
Sol: $r_2 = 1$ and $r_1 = 1 - cos\theta$. From Fig, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

$$A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} ({r_2}^2 - {r_1}^2) \, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} (1 - (1 - cos\theta)^2) \, d\theta$$

$= 2 \int_0^{\frac{\pi}{2}} \frac{1}{2} (2cos\theta - cos^2\theta) \, d\theta = \int_0^{\frac{\pi}{2}} (2cos\theta - \frac{(1+cos2\theta)}{2}) \, d\theta = \frac{8-\pi}{4}.$

$A = 2A_1.$

Ex: Find the area outside the Circle $r = \frac{a}{2}$ and inside the curve $r = acos2\theta$.



Sol: First, we find the intersection point. $\frac{a}{2} = acos2\theta$

$\Rightarrow \frac{1}{2} = cos2\theta$, $\Rightarrow 2\theta = \frac{\pi}{3}, -\frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{6}, -\frac{\pi}{6}$

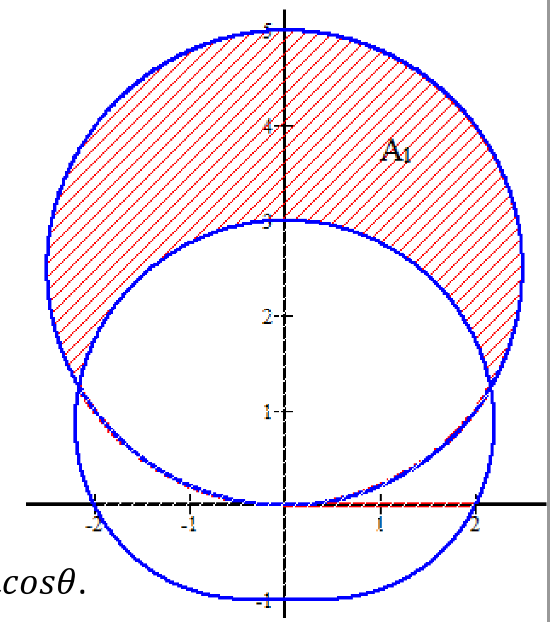
$$A = 8A_1 = 8\int_0^{\frac{\pi}{6}} \frac{1}{2}({r_2}^2 - {r_1}^2)d\theta$$

$$= 4\int_0^{\frac{\pi}{6}} \left(\left(\frac{a}{2}\right)^2 - (acos2\theta)^2\right)d\theta = a^2\left(\frac{\pi}{6} + \frac{\sqrt{3}}{4}\right)$$

Ex: Find the area inside the Circle $r = 5sin\theta$ and outside the curve $r = 2 + sin\theta$.

Sol: First, we find the intersection point $5sin\theta = 2 + sin\theta$

$\Rightarrow 4sin\theta = 2 \Rightarrow sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$.

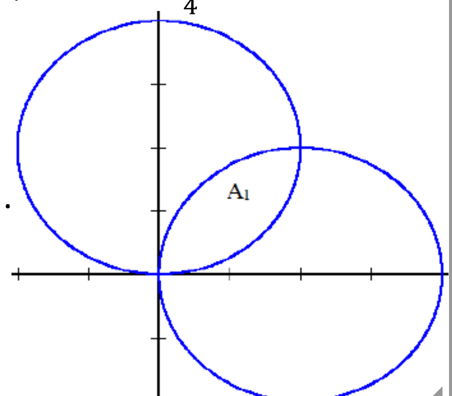


$$A = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}({r_2}^2 - {r_1}^2)d\theta = 2A_1$$

$$= 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{2}((5sin\theta)^2 - (2 + sin\theta)^2)d\theta = \frac{8\pi}{3} + \sqrt{3}$$

Ex: Find the area of the intersection $r = 2asin\theta$, $r = 2acos\theta$.

Sol: Find the intersection point, $2asin\theta = 2acos\theta \Rightarrow tan\theta = 1$, then $\theta = \frac{\pi}{4}$.

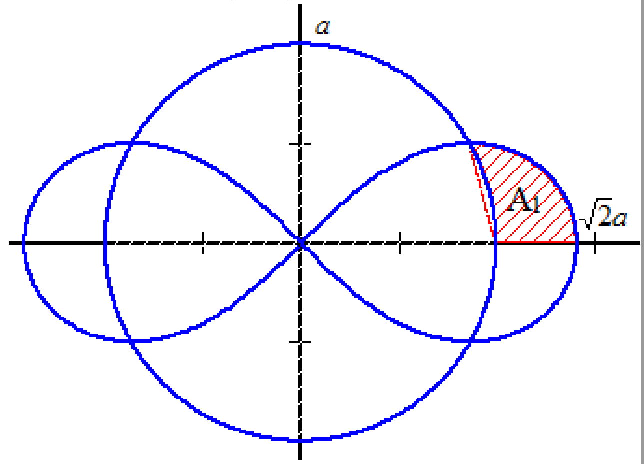


$$A_1 = \int_0^{\frac{\pi}{4}} \frac{1}{2}({r_2}^2 - {r_1}^2)d\theta = \int_0^{\frac{\pi}{4}} \frac{1}{2}((2acos\theta)^2 - (2asin\theta)^2)d\theta$$

$$= 2a^2\int_0^{\frac{\pi}{4}} \left(\left(\frac{1+cos2\theta}{2}\right) - \left(\frac{1-cos2\theta}{2}\right)\right)d\theta = 2a^2\int_0^{\frac{\pi}{4}} cos2\theta = a^2.$$

Ex: Find the area inside $r^2 = 2a^2cos2\theta$ and outside $r = a$.

Sol: intersection point, $2a^2cos2\theta = a^2 \Rightarrow cos2\theta = \frac{1}{2}$, then $\theta = \frac{\pi}{6}, \frac{-\pi}{6}$.

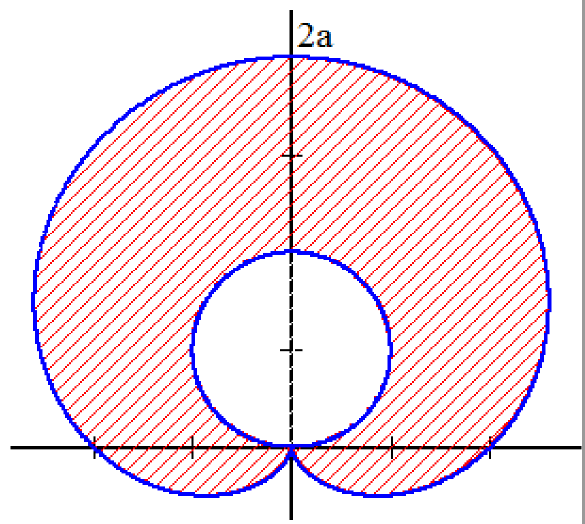


$$A = 4A_1 = 4\int_0^{\frac{\pi}{6}} \frac{1}{2}({r_2}^2 - {r_1}^2)d\theta$$

$$= 2\int_0^{\frac{\pi}{6}} \frac{1}{2}(2a^2cos2\theta - a^2)d\theta$$

$$= 4a^2\int_0^{\frac{\pi}{6}} \left(cos2\theta - \frac{1}{2}\right)d\theta = a^2\left(\sqrt{3} - \frac{\pi}{3}\right)$$

Ex: Find the area inside $r = a(1 + sin\theta)$ and outside $r = asin\theta$.



Sol: $A_1 = \pi\left(\frac{a}{2}\right)^2 = \frac{\pi}{4}a^2$.

$$A_2 = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}r^2d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a^2(1 + sin\theta)^2\, d\theta$$

$$= a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1 + 2sin\theta + \left(\frac{1 - cos2\theta}{2}\right)\right)d\theta$$

$$= a^2\left[\frac{3}{2}\theta - 2cos\theta - \frac{1}{2}\frac{sin2\theta}{2}\right]\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi}{2}a^2$$

Then $A = A_2 - A_1 = \frac{\pi}{4}a^2$.

 # Arc Length of a Polar Curve

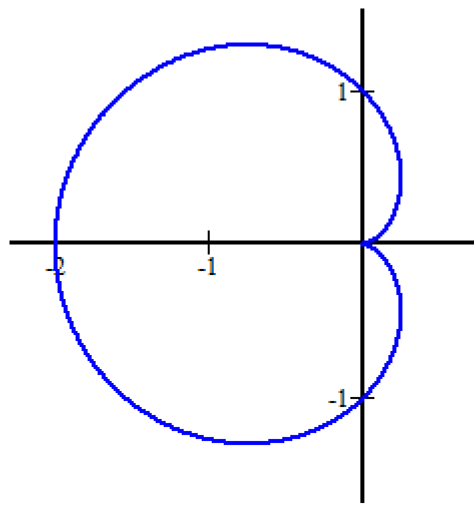
We can obtain a polar coordinate formula for the length of curve $r = f(\theta), \alpha \leq \theta \leq \beta$, by parameterizing the curve as $x = rcos\theta = f(\theta)cos\theta\ , y = rsin\theta = f(\theta)sin\theta$ $\alpha \leq \theta \leq \beta$. The parametric length formula equation, then gives

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \ \ \text{s.t}\ dx = -rsin\theta d\theta + cos\theta dr; dy = rcos\theta d\theta + sin\theta dr$$

This equation becomes

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \ \ \text{s.t}\ (ds)^2 = (rd\theta)^2 + (dr)^2$$

Ex: Find the length of the cardioid $r = 1 - cos\theta$.



Sol: $r = 1 - cos\theta \ \Rightarrow \frac{dr}{d\theta} = sin\theta$

$$(ds)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2 = (1 - cos\theta)^2 + (sin\theta)^2 = 2 - 2cos\theta$$

$$L = \int_0^{2\pi} \sqrt{2 - 2cos\theta}\, d\theta = \int_0^{2\pi} \sqrt{4\sin^2\left(\frac{\theta}{2}\right)}\, d\theta$$

$$= \int_0^{2\pi} 2\sin\left(\frac{\theta}{2}\right) d\theta = 8$$

Ex: Find the length of $r = a(1 + cos\theta)$. (H.W.).

Ex: Show that the point $(2a, \pi)$, contains in the intersection $r = a(1 - cos\theta)$d $r^2 = 4a^2cos\theta$.

Sol: Curve 1: $2a = a(1 + cos\pi) = a\big(1 - (-1)\big) = 2a \ \therefore$ contains in $r = a(1 - cos\theta)$.

Curve 2: $4a^2 = 4a^2cos\pi \ \Rightarrow \ 4a^2 = -4a^2$ (no right), so take clockwise point

$(-r, \theta - \pi)$, then $4a^2 = 4a^2(1) = 4a^2$. Satisfy 2.