1. Ring

A system $(R,+,\cdot)$*, where* R *is a non-empty set and addition* $(+)$*, multiplication* $(\cdot)$ *are two binary compositions on* R*, is called a ring if it is satisfies the following postulates:*

Under Addition

(i) **Closure Axiom.** $\forall a,b \in R \Rightarrow a+b \in R$

(ii) **Associative Law**. $a+(b+c)=(a+b)+c, \forall a,b,c \in R$

(iii) **Existence of Identity**. *There exists an element* $0 \in R$*, called* **the identity under addition**, *such that*

$$a+0=a=0+a, \forall a \in R$$

*[0 is also called the **zero-element**]*

(iv) **Existence of Inverse**. $\forall a \in R$*, there exists an element* $-a \in R$*, called* **the inverse of a under addition**, *such that*

$$a+(-a)=0=(-a)+a$$

(v) **Commutative Law.** $\forall a,b \in R, a+b=b+a.$

Under Multiplication

(vi) **Closure Axiom**. $\forall a,b \in R \Rightarrow a \cdot b \in R.$

(vii) **Associative Law**. $a \cdot (b \cdot c)=(a \cdot b)\cdot c, \forall a,b,c \in R.$

(viii) **Distributive Law.** $\forall a,b,c \in R,$

(I) $a \cdot (b+c)=a \cdot b + a \cdot c$

(II) $(b+c)\cdot a = b \cdot a + c \cdot a$

Conclusion

(i) **R forms an abelian group under addition.**

(ii) **R is a semi-group under multiplication.**

(iii) **R satisfies distributive laws.**

Note: *the ring is called* **non-associative** *if associative law under multiplication does not hold.*

2. **Commutative Ring or Abelian Ring**

   In addition to the above eight postulates, the following postulate is also satisfied, the ring R is called a commutative or an abelian ring

   (ix) **Commutative Law.** $\forall a, b \in R, a \cdot b = b \cdot a$

3. **Ring with Unity**

   A ring R which contains the multiplicative identity (called unity) is called a ring with unity.

   Thus if $1 \in R$ *such that* $a \cdot 1 = a = 1 \cdot a, \forall a \in R$, *then the ring is called a ring with unity.*

4. **Ring without Unity**

   a ring R which does not contain multiplicative identity is called a ring without unity.

5. **Finite and Infinite Ring**

   If the number of elements in the ring R is finite, then R is called a finite ring, otherwise it is called an infinite ring.

6. **Order of Ring**

   The number of elements in a finite ring is called the order of the ring. This is denoted by $O(R)$ *or* $| R |$.

7. **Units of a ring with unity**

   The elements which possess inverses under the second operation $(\cdot)$ *are called units of a ring.*

   In the set I of integers, we know that $(-1)(-1) = 1$.

   Thus -1 *is the* **unit** *but not* **unity** *of the ring.*

   Again $1 \cdot 1 = 1$

Thus 1 *is the* **unit** *as well as* **unity** *of the ring.*

Note: *unity is a unit but every unit is not a unity.*

8. **Zero divisors of a ring**

   Let $(R, +, \cdot)$ *be a ring*

   $\forall a, b \in R$, *where* $a \neq 0, b \neq 0$

   If $ab = 0$, *then* R *is called* **a ring with zero divisors**.

   Here a *is called the* **left-zero divisor** *and* b *is called the* **right-zero divisor.**

   An element which is left as well as right-zero divisors is called the **zero divisor of the ring.**

   In abelian rings, every left-zero divisor is also the right-zero divisor and vice-versa.

9. **Ring without zero-divisor**

   The ring which is not with zero divisor is called the ring without zero divisor.

   i.e. if $a \neq 0, b \neq 0$, *then* $ab \neq 0$

example 1. *Prove that* $(Z, +, \cdot)$, *where* Z *is a set of all integers, is a ring.*

Sol. *The system is* $(Z, +, \cdot)$, *where*

$$Z = \{\dots \dots - 3, -2, -1, 0, 1, 2, 3, \dots \dots\}$$

And '+' and '.' Are binary composition in Z.

Under addition

(i) **Closure axiom.** $\forall a, b \in Z \Longrightarrow a + b \in Z$. *[$\because$ sum of two integers is an integer]*

(ii) **Associative law.** $a + (b + c) = (a + b) + c, \forall a, b, c \in Z$

(iii) **Existence of identity.** *There exists an element* $0 \in Z$, *such that* $a + 0 = a = 0 + a, \forall a \in Z$.

(iv) **Existence of inverse.** $\forall a \in Z$, *there exists an element* $-a \in Z$, *such that* $a + (-a) = 0 = (-a) + a$

(v) **Commutative law.** $\forall a, b \in Z, a + b = b + a$.

Under multiplication

(i) **Closure axiom.** $\forall a, b \in Z \Longrightarrow a \cdot b \in Z$. *[$\because$ product of two integers is an integer]*

(ii) **Associative law.** $a \cdot (b \cdot c) = (a \cdot b) \cdot c, \forall a, b, c \in Z$.

(iii) **Distributive law.** $\forall a, b, c \in Z$,

(I) $a \cdot (b + c) = a \cdot b + a \cdot c$

(II) $(b + c) \cdot a = b \cdot a + c \cdot a$

Hence $(Z, +, \cdot)$ *is a ring.*