**Examples:**

1- *For* $n > 1$ *then* $(\boldsymbol{nZ}, +, .)$ *is commutative ring but without identity.*

$$2Z = \{0, \mp 2, \mp 4, \mp 6, \dots \dots \dots\} = \langle 2 \rangle$$

2- *The square matrix* $(\boldsymbol{M_{n\times n}}, +, .)$ *Is a ring with identity but does not commutative. Why?*

3- $(\boldsymbol{Z_n}, \oplus, \otimes)$ *is a commutative ring with identity (set from 0 to n-1).*

4- *The set of natural numbers* $\boldsymbol{N}$ *is not a ring because it is not a group.*

5- $(\boldsymbol{Q}, +, .)$ *is a ring.*

**Example 2.** *Prove that the set* $R = \{0,1,2,3,4,5\}$ *is a commutative ring with respect to the operations of addition (mod 6) and multiplication (mod 6).*

**Sol.** *The composition* $(+)$ *table is*

| + | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 |
| 1 | 1 | 2 | 3 | 4 | 5 | 0 |
| 2 | 2 | 3 | 4 | 5 | 0 | 1 |
| 3 | 3 | 4 | 5 | 0 | 1 | 2 |
| 4 | 4 | 5 | 0 | 1 | 2 | 3 |
| 5 | 5 | 0 | 1 | 2 | 3 | 4 |

***(i)*** *Since all possible sums belong to R, therefore, R is closed w.r.t. addition (mod 6).*

**(ii)** *Associative law holds because* $a + (b + c) = (a + b) + c = a + b + c$ *under addition (mod 6)*

**(iii)** *Here 0 is the additive identity because* $a + 0 = a = 0 + a$ *(mod 6).*

**(iv)** *From the table we see that*

$(0)^{-1} = 0, (1)^{-1} = 5, (2)^{-1} = 4, (3)^{-1} = 3, (4)^{-1} = 2, (5)^{-1} = 1$

$\therefore$ *the inverse of every element exists.*

**(v)** *Commutative law holds because* $a + b = b + a$ *(mod 6).*

*The composition* $(\times)$ *table is*

| + | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | 4 | 5 |
| 2 | 0 | 2 | 4 | 0 | 2 | 4 |
| 3 | 0 | 3 | 0 | 3 | 0 | 3 |
| 4 | 0 | 4 | 2 | 0 | 4 | 2 |
| 5 | 0 | 5 | 4 | 3 | 2 | 1 |

**(vi)** *From the composition table it is clear that R is closed for multiplication.*

**(vii)** *Associative law holds because* $a \times (b \times c) = (a \times b) \times c = a \times b \times c$ *under multiplication (mod 6).*

**(viii)** $a \times (b + c) = a \times [b + c]$

*Where* $[b + c]$ *is the least non-negative remainder obtained when* $b + c$ *is divided by* 6

$$= [a(b + c)] \; reduced \; modulo \; 6$$

$$= [ab + ac]$$

$$= [ab] + [ac]$$

$$= (a \times b) + (a \times c)$$

*Thus R is a ring.*

***(ix)*** *Commutative law holds because* $a.b = b.a$ *(mod 6)*

*Hence R is a commutative ring.*