PROPERTIES OF RINGS

Let $(R,+,.)$ *be a ring, then* $\forall a,b \in R$,

(i) $a.0+0.a=0$, *where* 0 *is additive identity.*

(ii) $a(-b)=-(ab)=(-a)b$.

(iii) $(-a)(-b)=ab$

Proof. (i) $\forall a \in R$,

$a.0=a.(0+0)=a.0+a.0$

[*By Right Distributive Law*]

$\Longrightarrow 0=a.0$ [*By right cancellation since*

R is a group under addition]

$\Longrightarrow a.0=0$

Similarly $0.a=0$

Hence $a.0=0.a=0$

(ii) $a[b+(-b)]=a.0=0$ [*By part* (i)]

$\Longrightarrow ab+a(-b)=0$

$\Longrightarrow a(-b)=-(ab)$

Similarly $(-a)b=-(ab)$

Hence $a(-b)=-(ab)=(-a)b$

(iii) $(-a)(-b)=-[(-a)b]=-[1-(ab)]$

$=$ *inverse of* (*inverse of* ab)

$=ab$

Example 9. *If R is a ring with unity element 1, then*

(i) $(-1).a = -a = a.(-1)$

(ii) $(-1).(-1) = 1$

Sol. *(i)* $[1 + (-1)].a = 1.a + (-1).a$ $[Right\ Distributive\ Law]$

$$\Rightarrow 0.a = a + (-1).a$$

$$\Rightarrow 0 = a + (-1).a$$

$$\therefore (-1).a = -a$$

Similarly $a.(-1) = -a$.

(iii) $(-1).(-1) = -(-1)$ $[By\ part\ (i)]$

$$= inverse\ of\ (inverse\ of\ 1) = 1.$$

CANCELLATION LAWS IN A RING

Let $(R, +, .)$ be a ring . since $(R, +, .)$ Is an abelian group, therefore, cancellation laws for addition hold good. We also say that the cancellation laws for multiplication also hold good iff $\forall a, b, c \in R$, we have

$$a \neq 0, ab = ac \Rightarrow b = c\ [Left\ Cancellation\ Law]$$

and $a \neq 0, ba = ca \Rightarrow b = c\ [Right\ Cancellation\ Law]$

Theorem. *Prove that a ring R is without zero divisors iff the cancellation laws hold in R.*

proof. *Let R be a ring without zero divisors.*

Let $a, b, c \in R$ be three elements such that

$a.b = a.c$ *where* $a \neq 0$

$\Rightarrow a.b - a.c = 0$

$\Rightarrow a.(b - c) = 0$

Since the ring is without zero divisors and $a \neq 0$

$\therefore a.(b - c) = 0 \Rightarrow b - c = 0 \Rightarrow b == c$

Thus $a.b = a.c \Rightarrow b = c$

i.e., left cancellation law holds good.

Similarly, right cancellation law holds good.

Conversely. *Let us suppose that cancellation laws hold good in R.*

Let, if possible $a.b = 0$ *where* $a \neq 0, b \neq 0$.

Then $a.b = 0 \Rightarrow a.b = a.0 \Rightarrow b = 0$ $[By\ Left\ Cancellation\ Law]$

Which contradicts the assumption

$\therefore a.b = 0 \Rightarrow either\ a = 0\ or\ b = 0$

$or\ a = 0\ and\ b = 0$

i.e. R is without zero divisors.