DIVISION RING, INTEGRAL DOMAIN AND FIELD

Division Ring. *A ring with at least two elements is said to be a division ring if its non-zero elements form a group w.r.t multiplication composition.*

This is also known as **skew field.**

Remember: a ring R is a division ring if

(I) R has at least two elements.

(II) R has unity.

(III) Each non-zero element of R has multiplicative inverse.

Integral Domain. *A commutative ring with unity element containing at least two elements is an integral domain if it is containing no proper zero divisors.*

Remember. A ring R is an integral domain if

(I) R is commutative

(II) R has a unity element.

(III) R is without zero divisors.

Illustrations:

(I) Ring of integers is an integral domain.

(II) Ring of numbers $a + bi$, *where* $a, b \in Z$, *is an integral domain.*

(III) Ring of numbers $a + b\sqrt{2}$, *where* $a, b \in Z$ *is an integral domain.*

(IV) The ring of even integers with zero is not an integral domain since it does not contain the unity element 1 *such that* $a.1 = 1.a = a$ *though it does not have zero divisors.*

Field: *A ring R is said to be a field if it has at least two elements and (i) is commutative (ii) has unity (iii) every non-zero element of R is invertible w.r.t multiplication.*

Illustrations:

(I) *Ring of rational numbers, ring of real numbers and ring of complex numbers are fields.*

(II) *Each commutative division ring is a field.*

(III) *Set of integers under addition* $\equiv (mod\ 5)$ *and multiplication* $\equiv (mod\ 5)$ *is a finite field having* 5 *elements* $[\{0\},\{1\},\{2\},\{3\},\{4\}]$
inverse of $\{2\}$ *is* $\{3\}$ *because* $\{2\}\times\{3\}=\{1\}$
Inverse of $\{1\}$ *is* $\{1\}$
Inverse of $\{4\}$ *is* $\{4\}$.

(IV) *Set of integers under addition* $\equiv (mod\ 6)$ *and multiplication* $\equiv (mod\ 6)$ *is not a field but a ring.*

(V) *Set of integers under addition* $\equiv (mod\ p)$ *and multiplication* $\equiv (mod\ p)$ *is a field if and only if p is prime.*

Theorem. *A finite integral domain* D *is a field.*

Proof. Homework

Theorem. *Every field* F *is an integral domain.*

Proof: *Here we are to show that a field has no zero divisors.*

Let $a, b \in F$ *with* $a \neq 0$ *such that* $ab = 0$

Since $a \neq 0, \ \therefore \ a^{-1}$ *exists.*

Now $\quad ab = 0$

$\Longrightarrow \quad a^{-1}(ab) = a^{-1}0$

$\Longrightarrow \quad (a^{-1}a)b = 0$

$\Longrightarrow \quad b = 0$

Similarly $\quad ab = 0$ *with* $b \neq 0 \Longrightarrow a = 0$

Thus F *has no zero divisors.*

Hence every field is an integral domain.

Idempotent element

Let $(R, +, .)$ *be a ring, an element* $a \in R$ *is said to be an idempotent if*

$$a^2 = a \Longrightarrow a.a = a$$

Example. *In* Z, Q, R *idempotent elements are* $0,1$.

Example. *In* $2Z$, *idempotent element is* 0.

Nilpotent element

Let $(R, +, .)$ *be a ring. An element* $a \in R$ *is said to be nilpotent if*

$$a^n = 0, n > 0.$$

Example. *In* Z, Q, R *nilpotent element is* 0.

Unit element

Let $(R, +, .)$ *be a ring with identity* 1. *An element* $a \in R$ *is said to be unit in* R *if there exists* $b \in R$ *such that* $ab = ba = 1$.

Example. *In Z the units are* $-1,1$.

Example. *In Q, R every non-zero element is unit.*

Boolean Ring

A ring in which every element is an idempotent is said to be Boolean ring
i.e. $a^2 = a, \forall a \in R \Longrightarrow (R, +, .)$ *is Boolean Ring*