Conclusions.

(I) Every field is a division ring but not vice-versa.

(II) Every field is an integral domain but not vice-versa.

(III) A field is a commutative division ring.

Example 10. *(i) Give an example of a division ring which is not a field.*

(ii) Give an example of an integral domain which is not a field

Sol. *(i) The set of matrices*

$\begin{bmatrix} a+ib & c+id \\ -c+id & a-ib \end{bmatrix}$, *where* a, b, c, d *are real numbers is a division ring which is not a field. [$\because$ Matrix multiplication is not, in general, commutative].*

(ii) The ring of integers is an integral domain which is not a field because all the elements do not have multiplicative inverses.

Example 11. *Show that the commutative ring D is an integral domain if and only if for $a, b, c \in D$, with $a \neq 0$ the relation $ab = ac \Rightarrow b = c$.*

Sol. *Let D be an integral domain*

i.e. D has no zero divisors.

we have:

$$ab = ac \quad \Rightarrow \quad ab - ac = 0$$

$$\Rightarrow \quad a(b-c) = 0 \quad \Rightarrow \quad b - c = 0$$

$$[\because a \neq 0 \text{ and } D \text{ is without zero divisor}]$$

$$\Rightarrow \quad b = 0$$

Conversely. *let* $ab = ac \Longrightarrow b = c$

If possible, let $ab = 0$, *where* $a \neq 0, b \neq 0$.

Then we have $ab = a.0 \qquad [\because a.0 = 0]$

$\Longrightarrow \qquad b = 0 \qquad [Left\ Cancellation\ Law]$

which is contradiction.

Hence D *is without zero divisor i.e.,* D *is an integral domain.*

CHARACTERISTIC OF A RING

If 0 *denotes the zero-element of a ring* R *and suppose there exists a positive integer* n *such that*

$$na = a + a + \cdots + a = 0 \ \ \forall a \in R$$

The smallest such positive integer n *is called the characteristic of the ring* R.

if such an integer n *does not exist, the ring is said to have* **characteristic zero or infinite.**

All the rings have characteristic zero.

The ring of integers modulus 6 *has characteristic* 6 *since*

$$\{1\} + \{1\} + \{1\} + \{1\} + \{1\} + \{1\} = \{6\} = \{0\}$$

Example 12. *Let a, b be commutative elements of a ring R of characteristic two. Show that*

$$(a+b)^2 = a^2 + b^2 = (a-b)^2$$

Sol. *We have:* $\quad ab = ba \quad for \quad a, b \in R$

since characteristic of R is two,

$$\therefore x + x = 0 \quad \forall x \in R$$

Now $\quad (a+b)^2 = (a+b)(a+b)$

$$= aa + ab + ba + bb$$

$$= a^2 + ab + ba + b^2$$

$$= a^2 + x + x + b^2 \quad [Let\ ab = ba = x]$$

$$= a^2 + b^2$$

similarly $\quad (a-b)^2 = (a-b)(a-b)$

$$= a.a + a(-b) + (-b)a + (-b)(-b)$$

$$= a^2 - ab - ba + b^2$$

$$= a^2 - (ab + ba) + b^2$$

$$= a^2 - (x + x) + b^2 \quad [x + x = 0]$$

$$= a^2 + b^2.$$

Example 13. *The characteristic of an integral domain is either zero or a prime number.*

Sol. *Let D be an integral domain.*

Let $a \in D$, where $a \neq 0$

If $O(a) = 0$, then characteristic of D is zero.

If $O(a)$ is finite, then characteristic of D is p.

To prove. *p is prime*

Suppose that p is not prime.

Then $p = p_1 p_2$, where $p_1 \neq 1, p_2 \neq 1$ and $p_1, p_2 < p$.

Since D is an integral domain, $\therefore\ a \neq 0$

$\Longrightarrow \quad aa \neq 0 \quad \Longrightarrow \quad a^2 \neq 0$

Now $\quad O(a) = p \quad \Longrightarrow \quad O(a^2) = p$

$\Longrightarrow \quad pa^2 = 0 \quad \Longrightarrow \quad (p_1 p_2)a^2 = 0$

$\Longrightarrow \quad (a^2 + a^2 + \cdots \text{to } p_1 p_2 \text{ terms}) = 0$

$\Longrightarrow (p_1 a)(p_2 a) = 0$

$\Longrightarrow$ *either* $\quad p_1 a = 0 \quad$ *or* $\quad p_2 a = 0$

[*D is without zero divisor*]

But $\quad p_1 < p \quad$ *and* $\quad p_2 < p$

Also p is the least positive integer such that $pa = 0$.

Hence p is prime.