Proof of equation (1). We know that the following identity is true.

$$1 - max[\mu_A, \mu_B] = min[1 - \mu_A, 1 - \mu_B] \cdots\cdots\cdots\cdots (3)$$

To show that we consider the two possible cases: $\mu_A \geq \mu_B$ and $\mu_A < \mu_B$. If $\mu_A \geq \mu_B$, then $1 - \mu_A \leq 1 - \mu_B$ and $1 - max[\mu_A, \mu_B] = 1 - \mu_A = min[1 - \mu_A, 1 - \mu_B]$, which is equation (3). If $\mu_A < \mu_B$, then $1 - \mu_A > 1 - \mu_B$ and $1 - max[\mu_A, \mu_B] = 1 - \mu_B = min[1 - \mu_A, 1 - \mu_B]$ which is again equation (3). Hence this equation (3) is true. Now, the membership function of $(A \cup B)^c$ is given by

$$\begin{aligned}
\mu_{(A \cup B)^c}(x) &= 1 - \mu_{A \cup B}(x) \\
&= 1 - max[\mu_A(x), \mu_B(x)] \\
&= min[1 - \mu_A(x), 1 - \mu_B(x)] \\
&= min[\mu_{A^c}(x), \mu_{B^c}(x)] \\
&= \mu_{A^c \cap B^c}(x)
\end{aligned}$$

This proves Equ. (1). Similarly, using (3) we can prove Equ. (2).

1.3 Images and Preimages of Fuzzy Sets

Definition 1.3.1 [6]: The symbol I will denote the unit interval [0,1]. Let X be a non-empty set. Now, for the sake of simplicity of notation we will not differentiate between A and μ_A. That is a fuzzy set A in X is a function with domain X and values in I, i.e. an element of I^X. Let $A, B \in I^X$ and let $f : X \to Y$ be a function. Then $f(A) \in I^Y$, i.e. $f(A)$ is a fuzzy set in Y, defined by

$$f(A)(y) = \begin{cases} \sup\{A(x) : x \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{if } f^{-1}(y) = \phi, \end{cases}$$

and $f^{-1}(B)$ is a fuzzy set in X, defined by $f^{-1}(B)(x) = B(f(x))$, $x \in X$.

Definition 1.3.2 [6]. The product $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ of mapping $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ is defined by $(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2))$ for each $(x_1, x_2) \in X_1 \times X_2$. For a mapping $f : X \to Y$, the graph $g : X \to X \times Y$ of f is defined by $g(x) = (x, f(x))$, for each $x \in X$.

Definition 1.3.3 [6]. Let $A \in I^X$ and $B \in I^Y$. Then by $A \times B$ we denote the fuzzy set in $X \times Y$ for which $(A \times B)(x, y) = min(A(x), B(y))$, for every $(x, y) \in X \times Y$.

Proposition 1.3.4 [6]. $f^{-1}(B^c) = (f^{-1}(B))^c$, for any fuzzy set B in Y.

Proof. $f^{-1}(B^c)(x) = (B^c)f(x) = 1 - B(f(x)) = 1 - f^{-1}(B(x)) = (f^{-1}(B))^c(x)$, $\forall x \in X$.

Proposition 1.3.5 [6]. $f(f^{-1}(B)) \leq B$, for any fuzzy set B in Y.

Proof. The proof follows by noting that

$$f(f^{-1}(B)(y)) = \begin{cases} \sup\{f^{-1}(B)(x) : x \in f^{-1}(y)\}, & \text{if } f^{-1}(y) \neq \phi \\ 0, & \text{if } f^{-1}(y) = \phi \end{cases}$$

$$= \begin{cases} \sup\{B(f(x)) : x \in f^{-1}(y)\}, & \text{if } f^{-1}(y) \neq \phi \\ 0, & \text{if } f^{-1}(y) = \phi \end{cases}$$

$$= \begin{cases} B(y), & \text{if } f^{-1}(y) \neq \phi \\ 0, & \text{if } f^{-1}(y) = \phi \end{cases}$$

Proposition 1.3.6 [6]. Let $f : X \to Y$ be a mapping and A_j be a family of fuzzy sets of Y, then

(a) $f^{-1}(\vee A_j) = \vee f^{-1}(A_j)$

(b) $f^{-1}(\wedge A_j) = \wedge f^{-1}(A_j)$

Proof-(a).

$$\begin{aligned} f^{-1}(\vee A_j)(x) &= (\vee A_j)(f(x)) \\ &= (A_1 \vee A_2 \vee \cdots \vee A_j \cdots)(f(x)) \\ &= max\{A_1 f(x), A_2 f(x), \cdots, A_j f(x) \cdots\} \\ &= max\{f^{-1}(A_1)(x), f^{-1}(A_2)(x), \cdots, f^{-1}(A_j)(x), \cdots\} \\ &= (f^{-1}(A_1) \vee f^{-1}(A_2) \vee \cdots \vee f^{-1}(A_j) \cdots)(x) \\ &= \vee f^{-1}(A_j)(x) \end{aligned}$$

(b).

$$\begin{aligned} f^{-1}(\wedge A_j)(x) &= (\wedge A_j)(f(x)) \\ &= (A_1 \wedge A_2 \wedge \cdots \wedge A_j \cdots)(f(x)) \\ &= min\{A_1 f(x), A_2 f(x), \cdots, A_j f(x) \cdots\} \\ &= min\{f^{-1}(A_1)(x), f^{-1}(A_2)(x), \cdots, f^{-1}(A_j)(x), \cdots\} \\ &= (f^{-1}(A_1) \wedge f^{-1}(A_2) \wedge \cdots \wedge f^{-1}(A_j) \cdots)(x) \\ &= \wedge f^{-1}(A_j)(x) \end{aligned}$$

Proposition 1.3.7 [6]. If A is a fuzzy set of X and B is a fuzzy set of Y, then $1-(A\times B) = (A^c \times 1) \vee (1 \times B^c)$.

Proof. $(1-(A\times B))(x,y) = max(1-A(x), 1-B(y)) = max((A^c\times 1)(x,y), (1\times B^c)(x,y)) = ((A^c \times 1) \vee (1 \times B^c))(x,y)$ for each $(x,y) \in X \times Y$.