Let $A, B \in I^X$. Then $A = B$ if and only if $P \in A \Leftrightarrow P \in B$ for every fuzzy point P in X.

Proposition 2.2.7 [6]. Let $\{A_j : j \in J\}$ be a family of fuzzy sets in X, P_x^a and P_y^b be fuzzy points in X and f be a map of X into Y. Then we have the following:

1. $P_x^a \in \vee\{A_j : j \in J\}$ if and only if there exists $j \in J$ such that $P_x^a \in A_j$.

2. If $P_x^a \in \wedge\{A_j : j \in J\}$, then for every $j \in J$ we have $P_x^a \in A_j$.

3. $P_x^a \in P_y^b$ if and only if $x = y$ and $a < b$.

4. If $P_x^a \in P_y^b$ and for every $j \in J, P_y^b \in A_j$, then $P_x^a \in \wedge\{A_j : j \in J\}$.

5. If $P_x^a \in A$, where A is a fuzzy set in X, then there exists $a < b$ such that $P_x^b \in A$.

6. $f(P_x^a) = P_{f(x)}^a$

7. $f((P_x^a)^c) = (f(P_x^a))^c$

8. If $P_x^a \in A$, then $f(P_x^a) \in f(A)$

9. If $P_x^a \in f^{-1}(B)$, then $P_{f(x)}^a \in B$, where B is a fuzzy set in Y.

10. If $P_y^b \in f(A)$, then there exists $x \in X$ such that $f(x) = y$ and $P_x^a \in A$.

11. If $P_y^b \in B$ and $y \in f(X)$, then for every $x \in f^{-1}(y)$ we have $P_x^b \in f^{-1}(B)$.

Proof. (1) $P_x^a \in \vee\{A_j : j \in J\}$ if and only if there exists $j \in J$ such that $P_x^a \in A_j$.

Let $P_x^a \in A_j \Rightarrow a \leq A_j(x) \Rightarrow a \leq max\{A_j(x) : j \in J\} \Rightarrow a \leq (\vee A_j)(x)$.

Again $P_x^a \in \vee\{A_j : j \in J\} \Rightarrow a \leq (\vee_{j \in J} A_j)(x) \Rightarrow a \leq A_j(x) \Rightarrow P_x^a \in A_j, j \in J$.

(4) If $P_x^a \in P_y^b$ and for every $j \in J, P_y^b \in A_j$, then $P_x^a \in \wedge\{A_j : j \in J\}$.

$P_x^a \in P_y^b \in A_j \Rightarrow P_x^a \in A_j \Rightarrow a \leq A_j(x) \Rightarrow a \leq min\{A_j(x) : j \in J\} \Rightarrow a \leq (\wedge A_j)(x) \Rightarrow$

$P_x^a \in \wedge\{A_j : j \in J\}$.

(6) $f(P_x^a) = P_{f(x)}^a$

$$f(P_x^a)(y) = \begin{cases} \sup\{P_x^a(z) : z \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{if } f^{-1}(y) = \phi, \end{cases}$$

$$= \begin{cases} a & \text{if } x \in f^{-1}(y), \\ 0 & \text{otherwise;} \end{cases}$$

$$= \begin{cases} a & \text{if } f(x) = y, \\ 0 & \text{otherwise;} \end{cases}$$

$= P_{f(x)}^a(y)\ \forall y \in Y \Rightarrow f(P_x^a) = P_{f(x)}^a$

(7) $f((P_x^a)^c) = (f(P_x^a))^c$

$f((P_x^a)^c)(y)=$

$$(P_{f(x)}^a)^c(y) = \begin{cases} 1-a & \text{if } y = f(x), \\ 0 & \text{otherwise;} \quad \cdots(i) \end{cases}$$

Now

$$(P_x^a)^c(z) = \begin{cases} 1-a & \text{if } z = x, \\ 0 & \text{if } z \neq x; \end{cases}$$

So

$$f((P_x^a)^c)(y) = \begin{cases} \sup\{(P_x^a)^c(z) : z \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{if } f^{-1}(y) = \phi, \end{cases}$$

$$= \begin{cases} 1-a & \text{if } x \in f^{-1}(y), \\ 0 & \text{otherwise}; \end{cases}$$

$$= \begin{cases} 1-a & \text{if } f(x) = y, \\ 0 & \text{otherwise}; \quad \cdots (ii) \end{cases}$$

Thus $f((P_x^a)^c) = (f(P_x^a))^c$.

Theorem 2.2.8 [8]. $\mathcal{B}$ is a base for an fts (X, τ) iff $\forall A \in \tau$ and for every fuzzy point P in A, $\exists B \in \mathcal{B}$ such that $P \in B \subseteq A$.

Proof. Assume that $\mathcal{B}$ is a base for τ, that is, every $A \in \tau$ is a union of members of B. Let $A \in \tau$ and $P_x^\alpha \in A$. So $A \in \tau \Rightarrow A = \bigcup_{i \in I}\{B_i : B_i \in \mathcal{B}\} \Rightarrow P_x^\alpha \in A = \bigcup_{i \in I}\{B_i : B_i \in \mathcal{B}\} \Rightarrow P_x^\alpha \in \bigcup_{i \in I}\{B_i : B_i \in \mathcal{B}\} \Rightarrow P_x^\alpha \in B_x \subseteq A$ (for some B_x).

Conversely, assume that for each $A \in \tau$ and for each $P_x^\alpha \in A$, $\exists\, B_x$ such that $P_x^\alpha \in B_x \subset A$. Let $A \in \tau$. To prove that A can be written as a union of members of $\mathcal{B}$ consider any arbitrary $P_x^\alpha \in A$. So by hypothesis $\exists\, B_x \in \mathcal{B}$ such that $P_x^\alpha \in B_x \subset A \Rightarrow A \subset \bigcup_{P_x^\alpha \in A} B_x$. Since $B_x \subset A$, for each $P_x^\alpha \in A$, therefore $A = \bigcup_{P_x^\alpha \in A} B$.

2.3 Closure and Interior of fuzzy sets

Definition 2.3.1 [6]. The closure $\overline{A}$ and the interior A^o of a fuzzy set A of X are defined as

$$\overline{A} = inf\{K : A \leq K, K^c \in \tau\}$$

$$A^o = sup\{O : O \leq A, O \in \tau\}$$

respectively.

Example 2.3.2 [6]. Let A, B and C be fuzzy sets of I defined as

$$A(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 2x-1 & \text{if } \frac{1}{2} \leq x \leq 1; \end{cases}$$

$$B(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq \frac{1}{4}, \\ -4x+2 & \text{if } \frac{1}{4} \leq x \leq \frac{1}{2}, \\ 0 & \text{if } \frac{1}{2} \leq x \leq 1; \end{cases}$$

$$C(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{4}, \\ \frac{4x-1}{3} & \text{if } \frac{1}{4} \leq x \leq 1; \end{cases}$$

Then $\tau = \{\overline{0}, A, B, A \vee B, \overline{1}\}$ is a fuzzy topology on I. It can be easily seen that $Cl(A) = B^c$, $Cl(B) = A^c$, $Cl(A \vee B) = \overline{1}$, $Int(A^c) = B$, $Int(B^c) = A$ and $Int(A \vee B)^c = \overline{0}$.

2.4 Neighborhood

Definition 2.4.1 [6]. A fuzzy point P_x^λ is said to be quasi-coincident with A, denoted by $P_x^\lambda qA$, if and only if $\lambda > A^c(x)$, or $\lambda + A(x) > 1$.

Proposition 2.4.2 [6]. Let f be a function from X to Y. Let P be a fuzzy point of X, A be a fuzzy set in X and B be a fuzzy set in Y. Then we have:

1 If $f(P)qB$, then $Pqf^{-1}(B)$.

2 If PqA, then $f(P)qf(A)$.

3 $P \in f^{-1}(B)$, if $f(P) \in B$.

4 $f(P) \in f(A)$, if $P \in A$.

Proof. (1) Let $P \equiv P_x^{\alpha}$, then

$$f(P_x^{\alpha})(y) = \begin{cases} \sup\{P_x^{\alpha}(z) : z \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{if } f^{-1}(y) = \phi; \end{cases}$$

$$= \begin{cases} 0 & \text{if } f^{-1}(y) = \phi \\ \alpha & \text{if } x \in f^{-1}(y), \text{if} f(x) = y \\ 0 & \text{if } x \notin f^{-1}(y); \end{cases}$$

Now, $f(P_x^{\alpha}) \equiv P_{f(x)}^{\alpha} \Rightarrow f(P_x^{\alpha})qB = P_{f(x)}^{\alpha}qB$.

Note that $P_{f(x)}^{\alpha}qB \Rightarrow \alpha + B(f(x)) > 1 \Rightarrow \alpha + f^{-1}B(x) > 1 \Rightarrow P_x^{\alpha}qf^{-1}(B)$, which completes the proof.

Definition 2.4.3 [6]. A fuzzy set A in (X, τ) is called a neighborhood of fuzzy point P_x^{λ} if and only if there exists a $B \in \tau$ such that $P_x^{\lambda} \in B \leq A$; a neighborhood A is said to be open if and only if A is open. The family consisting of all the neighborhoods of P_x^{λ} is called the system of neighborhoods of P_x^{λ}.

Definition 2.4.4 [6]. A fuzzy set A in (X, τ) is called a Q-neighborhood of fuzzy point P_x^{λ} if and only if there exists a $B \in \tau$ such that $P_x^{\lambda}qB \leq A$. The family consisting of all the Q-neighborhoods of P_x^{λ} is called the system of Q-neighborhoods of P_x^{λ}.

Proposition 2.4.5 [6]. $A \leq B$ if and only if A and B^c are not quasi-coincident; particularly, $P_x^{\lambda} \in A$ if and only if P_x^{λ} is not quasi-coincident with A^c.

Proof. $A(x) \leq B(x) \Leftrightarrow A(x) + B^c(x) = A(x) + 1 - B(x) \leq 1$. In particular $P_x^{\lambda} \in A \Rightarrow \lambda \leq A(x) \Rightarrow \lambda + A^c(x) \leq A(x) + A^c(x) \Rightarrow \lambda + A^c(x) \leq 1$.

Theorem-2.4.6 [6]. A fuzzy point $e \in A^o$ if and only if e has a neighborhood contained in A.