Chapter five

Numerical Differentiation and Integration

(1)If $x_{i+1} - x_i = h$

A. Numerical Differentiation of Newton Forward

$$f(x) = f(x_0) + (x - x_0)\frac{\Delta f_0}{1!\,h} + (x - x_0)(x - x_1)\frac{\Delta^2 f_0}{2!\,h^2} + \cdots + (x - x_0)(x - x_1)\ldots(x - x_n)\frac{\Delta^n f_0}{n!\,h^n} \ldots\ldots\ldots\ldots (1)$$

Let $\frac{x-x_0}{h} = q \Longrightarrow x - x_0 = hq.$

$x - x_1 = x - (x_0 + h) = x - x_0 - h = hq - h = h(q - 1)$

$x - x_2 = x - (x_1 + h) = x - x_0 - 2h = hq - 2h = h(q - 2)$

$x - x_3 = x - (x_2 + h) = x - x_0 - 3h = hq - 3h = h(q - 3)$

In general

$\boldsymbol{x - x_k = h(q - k) \quad , k = 0, 1, 2, 3, \ldots}$

Substitute the last relation in to (1) we obtain :

$$f(x) = f(x_0) + q\Delta f_0 + q(q - 1)\frac{\Delta^2 f_0}{2!} + \cdots + q(q - 1)\ldots(q - n)\frac{\Delta^n f_0}{n!}$$

$$f(x) = f(x_0) + q\Delta f_0 + \frac{1}{2}(q^2 - q)\Delta^2 f_0 + \frac{1}{6}(q^3 - 3q^2 - 2q)\Delta^3 f_0 + \cdots$$

$$f'(x) = \frac{df}{dq}.\frac{dq}{dx} = \frac{1}{h}.\frac{df}{dq}$$

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$$f'(x) = \frac{1}{h}\Big[\Delta f_0 + \frac{1}{2}(2q-1)\Delta^2 f_0 + \frac{1}{6}(3q^2 - 6q + 2)\Delta^3 f_0 + \frac{1}{24}(4q^3 - 18q^2 - 22q - 6)\Delta^4 f_0 + \cdots\Big] \ldots (2)$$

$$f''(x) = \frac{d^2 f}{dq^2} . \frac{d^2 f}{dx^2}$$

$$f''(x) = \frac{1}{h^2}\Big[\Delta^2 f_0 + (q-1)\Delta^3 f_0 + \frac{1}{12}(6q^2 - 18q + 11)\Delta^4 f_0 + \cdots\Big] \ldots (3)$$

If $x = x_i$

Set $x_i = x_0 \Longrightarrow q = 0\, , f(x_i) = f(x_0)$

$$f'(x_0) = \frac{1}{h}\Big[\Delta f_0 - \frac{1}{2}\Delta^2 f_0 + \frac{1}{3}\Delta^3 f_0 + \frac{1}{4}\Delta^4 f_0 + \cdots\Big] \ldots (4)$$

$$f''(x_0) = \frac{1}{h^2}\Big[\Delta^2 f_0 - \Delta^3 f_0 + \frac{11}{12}\Delta^4 f_0 - \frac{5}{6}\Delta^5 f_0 \ldots\Big] \ldots (5)$$