Remark 4.2:

For above example (4.2)

1- If $n = 1$ we get Ex.1.

2- If $n = 2$ we get Ex.4, (R^2, d) is called the Euclidean Plane.

3- (R^2, d) is called the Euclidean space. (الفضاء الإقليدي)

Basic Principles of Topology **مبادئ أساسية في التبولوجيا**

Definition 4.3:

Let (X, d) be a metric space, $x_\circ \in X$, and $0 < r \in R$, then

$B_r(x_\circ) = \{x \in X | d(x, x_\circ) < r\}$is called the ball whose center is $x_\circ$and radius is r.

$D_r(x_\circ) = \{x \in X | d(x, x_\circ) \leq r\}$is called the disc whose center is $x_\circ$and radius is r.

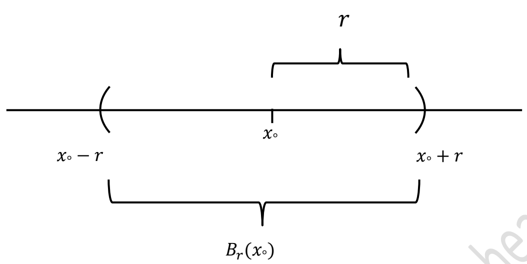
- It is clear that $x_\circ \in B_r(x_\circ)$ and $x_\circ \in D_r(x_\circ)$ (why?).
- In sometime we say that $B_r(x_\circ)$ is a neighborhood of $x_\circ$

Example 4.3:

1- Let (R,d) be the usual metric space and $d(x,y)=|x-y|$

$\therefore B_r(x_\circ=\{x\in R|d(x,x_\circ)<r\}=\{x\in R|\ |x-x_\circ|<r\}$

$=\{x\in R|x_\circ-r<x<x_\circ+r\}=(x_\circ-r,x_\circ+r)$



فترة مفتوحة في R

Also, $D_r(x_\circ)=[x_\circ-r,x_\circ+r]$

ملاحظة : إذن كل R هي عبارة عن فترة مفتوحة وكل قرص في R هو عبارة عن فترة مغلقة.
كرة في

ومن السهولة ان نلاحظ بان كل فترة مفتوحة R هي كرة وكل فترة مغلقة في R هي قرص .
في

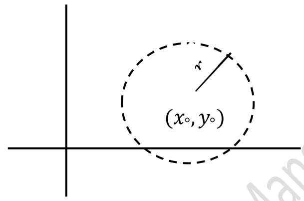
2- Let (R^2,d) be a metric space. Find $B_r((x_\circ,y_\circ))$

$$B_r\big((x_\circ,y_\circ)\big)=\{(x,y)\in R^2|d((x,y),(x_\circ,y_\circ))<r\}$$

$$= \left\{(x, y) \in R^2 \middle| \sqrt{(x - x_\circ)^2 + (y - y_\circ)^2} < r\right\}$$

$$= \{(x, y) \in R^2 | (x - x_\circ)^2 + (y - y_\circ)^2 < r^2\}$$

معادلة دائرة



$B_r\big((x_\circ, y_\circ)\big)$

3- Let (X, d) be the discrete metric

i.e., $d(x, y) = \begin{cases} 1 \text{ if } x \neq y \\ 0 \text{ if } x = y \end{cases} \quad \forall\, x, y \in X$. Then

$B_{\frac{1}{2}}(1) = \left\{x \in R \middle| d(x, 1) < \frac{1}{2}\right\} = \{1\}$. $\qquad B_{\frac{1}{2}}(2) = \{2\}$

$B_{10}(2) = R$, $\quad D_{\frac{1}{2}}(2) = \{2\}$, $\quad D_1(2) = R$, $\quad D_{10}(2) = R$.

Definition 4.4: (Interior point نقطة داخلية)

Let (X, d) be a metric space and $x_\circ \in S \subseteq X$, then we say that $x_\circ$ is an interior point of S if $\exists\, r > 0$ s.t $B_r(x_\circ) \subset S$.

Definition 4.5: (Open set) المجموعة المفتوحة

Let (X, d) be a metric space and $S \subseteq X$, then we say that S is an open set if $\forall x_\circ \in S \ \exists\, r > 0$ s.t $B_r(x_\circ) \subset S$.

i.e., S is an open set $\Leftrightarrow \forall x \in S, \ x$ is an interior point of S.

Proposition 4.2: Any ball is an open set

Proof:

Let $B = B_r(x_\circ)$ ball whose center is $x_\circ$ and radius r.

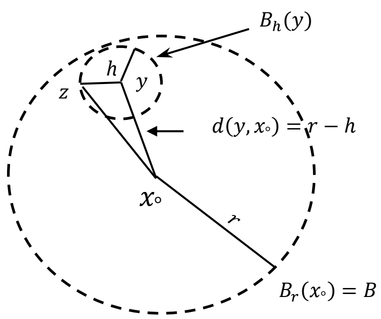
T.P B is an open set

i.e., T.P if $y \in B$, then $\exists\, h > 0$ (real number) s.t $B_h(y) \subset B$.

$\because y \in B \Rightarrow 0 \leq d(x_\circ, y) < r$

Let $h = r - d(x_\circ, y) > 0$

Let $z \in B_h(y)$. T.P $z \in B$, i.e., T.P $d(z, x_\circ) < r$



$$d(z, x_\circ) \leq d(z, y) + d(y, x_\circ)$$

$$< h + d(y, x_\circ)$$

$$= r - d(x_\circ, y) + d(y, x_\circ)$$

$$= r \ (d(x_\circ, y) = d(y, x_\circ))$$

$\Rightarrow z \in B_r(x_\circ) = B$

وهذا يعني كل نقطة من نقاط B هي نقطة داخلية في B .

Example 4.4:

From Ex.4.3 and Prop.4.2.

Any open interval (a, b) in R is an open set

$$x_\circ = \frac{a+b}{2} \ \forall^{\uparrow} r = \frac{d(a,b)}{2}$$

Also, in R^2 the set of all points inside of any circle is an open set.

Example 4.5:

1- $(a, \infty) = \{x \in R | x > a\}$ is an open set in R.

2- $(-\infty, a) = \{x \in R | x < a\}$ is an open set in R.

3- Q(rational numbers) is not open set in R. (why ?).

Let (R^2, d) be a metric space, then

4- $A = \{(x_1, x_2) \in R^2 | x_2 > 0\}$ is an open set in R^2.

5- $B = \{(x_1, x_2) \in R^2 | x_2 \geq 0\}$ is not open set in R^2. (why ?)

6- Every disc in R^2 $(or R^n)$ is not open in R^2 $(or R^n)$.

Definition 4.6 : (Boundary point) نقطة حدودية

Let (X, d) be a metric space and $S \subseteq X$, $x_\circ \in X$, then we say that $x_\circ$ is a boundary point of $S \Leftrightarrow \forall r > 0,\ B_r(x_\circ) \cap S \neq \emptyset\ \forall B_r(x_\circ) \cap S^c \neq \emptyset$ (where $S^c = X - S$).

Example 4.6 :

Let (R, d) be a metric space and let $S = [a, b]$ or $(a, b]$ or $[a, b)$. Then a and b are boundary points of S.

Theorem 4.1:

Let (X, d) be a metric space and let T be the family of all open subsets of X, then

1- $\emptyset \in T$ and $X \in T$.

2- The union of any number (finite or infinite) of open sets in T is also in T.

يعني اتحاد أي منظومة من المجموعات المفتوحة يكون مجموعة مفتوحة.

3- The intersection of a finite number of open sets in T is also in T.

يعني تقاطع أية منظومة منتهية من المجموعات المفتوحة يكون مجموعة مفتوحة.

i.e., Let $X \neq \emptyset$ be a metric space and let
$T = \{S | S \subseteq X$ and S open set in $X\}$. Then

1- $\emptyset \forall X \in T$

2- $\{S_\alpha\}_{\alpha\in\Lambda}$: $S_\alpha \in T\ \forall \alpha \in \Lambda \Rightarrow \bigcup_{\alpha\in\Lambda} S_\alpha \in T$. (Λ index set)

3- If $S_i \in T\ \forall_i$, $1 \le i \le n$ (n positive interger number) $\Rightarrow$ $\bigcap_{i=1}^{n} S_i \in T$.

Proof:

1- Let $x \in \emptyset \Rightarrow \exists B_r(x) \subset \emptyset \Rightarrow \emptyset$ is open $\Rightarrow \emptyset \in T$

$\because \nexists\ x \in \emptyset \Rightarrow \forall\ r > 0$, $B_r(x) \not\subset \emptyset \Rightarrow \emptyset$ is open

Also, $\forall x \in X$, $\exists\ r > 0$ s.t $B_r(x) \subset X \Rightarrow X$ is an open set.

2- Let $\{S_\alpha\}_{\alpha\in\Lambda}$ be a family of open sets, and let $x \in \bigcup_{\alpha\in\Lambda} S_\alpha$.

$\Rightarrow \exists \alpha_\circ \in \Lambda$ s.t $x \in S_{\alpha_\circ}$

$\because$ $S_{\alpha_\circ}$ is an open set $\Rightarrow x$ is an interior point of $S_{\alpha_\circ}$

$$\Rightarrow \exists B_r(x) \subseteq S_{\alpha_\circ} \subseteq \bigcup_{\alpha\in\Lambda} S_\alpha$$

$\Rightarrow x$ is an interior point of $\bigcup_{\alpha\in\Lambda} S_\alpha$

$\Rightarrow \bigcup_{\alpha\in\Lambda} S_\alpha$ is an open set.

3- Let $S_i, i = 1, 2, \cdots, n$ be open sets in T , and let $x \in \bigcap_{i=1}^{n} S_i \ \forall i$

$\Rightarrow x \in S_i \ \forall i \ (1 \le i \le n)$

$\because S_i$ is open set $\forall i \Rightarrow \exists r_i$ s.t $B_{r_i}(x) \subseteq S_i$

Let $0 < r = \min\{r_1, r_2, \cdots, r_n\}$

$\therefore \ B_r(x) \subseteq B_{r_i}(x) \forall i \subseteq S_i \ \forall i \ \subseteq \bigcap_{i=1}^{n} S_i$

$\therefore \bigcap_{i=1}^{n} S_i$ is an open set $\Rightarrow \bigcap_{i=1}^{n} S_i \in T$.

Remark 4.3.

Finiteness in (3) is essential

Let $G_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ $(n = 1, 2, \cdots)$. Then $G_n \forall n$ open in R^1, but

$\bigcap_{n=1}^{\infty} G_n = \{0\}$ which is not open in R^1.

Definition 4.7:

Let $X \neq \emptyset$ and T be the family of subsets of X which satisfy the conditions (1), (2) & (3) in the Theorem above (Th 4.1). Then T is called a topology on X and (X, T) is called a topological space.

Example 4.7:

1- Let $X \neq \emptyset$ be any set, and $T = \{\emptyset, X\}$, then (X, T) is a topological space.

This topology is called indiscrete topology or trivial topology.

2- Let $X \neq \emptyset$ be any set, and T be a family of all subsets of X, i.e., $T = \{A | A \subseteq X\}$, then (X, T) is a top.Space. This topology is called a discrete topology.

3- Let $X = \{a, b, c\}$, then

a- $T_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ is a topology on X

b- $T_2 = \{\emptyset, X, \{a, b\}, \{a, c\}\}$ is not a topology on X

because $\{a, b\} \cap \{a, c\} = \{a\} \notin T_2$.

c- $T_3 = \{\emptyset, X, \{a\}, \{b\}\}$ is not a topology on X. (why?)

ملاحظة: اذا كان (X, d) فضاء متري فيمكن استخدام البعد d لتعريف تبولوجيا T على X لجعل

(X, T) فضاء تبولوجيا . ان هذه العملية غير قابلة للانعكاس ، اي انه ليس كل تبولوجيا على

X يمكن ان يستحصل من تعريف بعد على X .

مثال : اذا كانت X = {a, b, c} و T = { $\emptyset, X$ } فان (X, T) فضاء تبولوجي . ولكن لا يوجد بعد

d على X يعطي هذا التبولوجيا.

Theorem 4.2:

S is an open set $\Leftrightarrow$ S is union of any number of balls.

Proof:

Suppose that S is an open set

$$\Rightarrow \forall\, x \in S\ \exists r_x > 0 \text{ s.t } B_{r_x}(x) \subset S.$$

$$\left.\begin{array}{r} \therefore \bigcup_{x\in S} B_{r_x}(x) \subset S \\ \text{but,} \qquad S \subseteq \bigcup_{x\in S} B_{r_x} \end{array}\right\} \Rightarrow S = \bigcup_{x\in S} B_{r_x}(x)$$

And , if $S = \bigcup_{x\in S} B_{r_x}(x) \Rightarrow S$ is an open set.

Definition 4.8:

Let X be a metric space (or topological space). A subset E of X is called closed $\Leftrightarrow E^c = X - E$ is open.

Example 4.8:

1- $[a, b]$ is a closed set in R, because
$R - [a, b] = (-\infty, a) \cup (b, \infty)$

2- $\{a, b\}$ is a closed set in R, because
$R - \{a, b\} = (-\infty, a) \cup (a, b) \cup (b, \infty)$

3- $(a, b]$ is not closed set in R, because
$R - (a, b] = (-\infty, a] \cup (b, \infty)$

4- Q is not closed in R. (why?)

5- In general any disc in R^n is closed set.

6- In general any ball in R^n is not closed set.

7- $[a, \infty) = \{x \in R | x \geq a\}$ is closed, because $R - [a, \infty) = (-\infty, a)$ open.

8- $(-\infty, b] = \{x \in R | x \leq b\}$ is closed, because $R - (-\infty, b] = (b, \infty)$ open.

Theorem 4.3:

Every finite set in a metric space (X, d) is closed.

Proof:

Let $A = \{x_1, x_2, \dots x_n\}$ and $A^c = X - A$. T.P A^c is open.

Let $y \in A^c \Rightarrow y \neq x_i \quad \forall i = 1, 2, \cdots, n \quad \Rightarrow d(y, x_i) = r_i > 0$

Let $r = \min\{r_1, r_2, \dots r_n\}$

$$\therefore B_r(y) \cap A = \emptyset \Rightarrow B_r(y) \subset A^c \Rightarrow A^c \text{ is open} \Rightarrow A \text{ is closed.}$$

Theorem 4.4:

Let X be a metric space (or a topological space) and T be the family of all closed subsets of X. i.e., $\tau = \{E \subseteq X | E \text{ is closed}\}$. Then.

1- $\emptyset, X \in \tau$

2- The union of a finite number of closed sets is closed.

3- The intersection of any number of closed sets is closed.

Proof: (check)

Remark 4.4:

The finiteness of the family in Th. 4.4.(2) is essential .i.e., The union of an infinite collection of closed set need not be closed.

Example 4.9:

Let $E_n = (-\infty, -\frac{1}{n}] \cup [\frac{1}{n}, \infty) \quad n = 1, 2, \cdots$ (closed)

$\Rightarrow E_n^c = \left(-\frac{1}{n}, \frac{1}{n}\right)$ open $\forall n = 1, 2, \cdots$

Then $\bigcup_{n=1}^{\infty} E_n$ is open (not closed) , because

$$\left(\bigcup_{n=1}^{\infty} E_n\right)^c = \bigcap_{n=1}^{\infty} E_n^c = \bigcap_{n=1}^{\infty}\left(-\frac{1}{n},\frac{1}{n}\right) = \{0\} \quad \text{closed}$$

ملاحظة : لاجل ان نفهم بصورة اوضح تكوين المجموعات المغلقة نعطي مفهوم نقطة التجمع (limit point) . إن هذا المفهوم يعتبر واحد من المفاهيم الاساسية في التحليل .