Chapter Five: The Continuity (الاستمرارية)

Definition 5.1: Let (X,d) and $(X`,d`)$ be a metric space. A mapping $f: X \to X`$ is called continuous at a point $x_0 \in X$ if for any $\epsilon > 0$, there is $\delta > 0$ (δ depends on ϵ and x_0) such that for each $x \in X$, if

$d(x,x_0) < \delta$, then $d`(f(x),f(x_0)) < \epsilon$.

Or

The $f: X \to X`$ is continuous at $x_0 \in X \Leftrightarrow$ for any ball $B`_\epsilon(f(x_0))$ with center $f(x_0)$ and radius ϵ in X, there is a ball $B_\delta(x_0)$ with center x_0 and radius δ in X such that

$$f(B_\delta(x_0)) \subseteq B`_\epsilon(f(x_0))$$

Remark 5.2: A mapping $f: X \to X`$ is called continuous (or continuous on X) iff f is continuous at each point in X.

Theorem 5.3: A mapping $f:(X,d) \to (X`,d`)$ is continuous iff for each open set U in $X`$, $f^{-1}(U)$ is open set on $X`$.

i.e.

$f: X \to X`$ is continuous $\Leftrightarrow$ $\forall U \subseteq_{open} X`$,

then,

$f^{-1}(U) \subseteq_{0pen} X$ where $f^{-1}(U) = \{x \in X / f(x) \in U\}$

Proof: $\Longrightarrow$

Suppose that f is continuous on X and $U \subseteq_{open} X`$

To prove $f^{-1}(U) \subseteq_{open} X$?

Let $x_0 \in f^{-1}(U)$

$$\Rightarrow f(x_0) \in U \overset{U \text{ is}}{\underset{open}{\Longrightarrow}} \exists \epsilon > 0 \text{ such that } B`_\epsilon(f(x_0)) \subset U$$

Since f is continuous $\Rightarrow \exists$ a ball $B_\delta(x_0)$ on X such that

$$f(B_\delta(x_0)) \subset B`_\epsilon(f(x_0)) \subset U$$

$$\Longrightarrow f\big(B_\delta(x_0)\big) \subset U \Longrightarrow B_\delta(x_0) \subset f^{-1}(U)$$

$$\Longrightarrow f^{-1}(U) \subseteq_{open} X. \blacksquare$$

$\Longleftarrow$

Suppose that, $\forall\, U \subseteq_{open} X`$, then,

$$f^{-1}(U) \subseteq_{open} X$$

To prove f is continuous?

Let $x_0 \in X$ and $B`_\epsilon\big(f(x_0)\big)$ be a ball in $X`$

$\Longrightarrow B`_\epsilon(f(x_0))$ is an open set in $X`$.

$\Longrightarrow f^{-1}(B`_\epsilon\big(f(x_0)\big))$ is an open set in $X$ and $x_0 \in f^{-1}(B`_\epsilon(f(x_0))$.

$\Longrightarrow \exists\ \delta > 0$ such that $B_\delta(x_0) \subset f^{-1}(B`_\epsilon(f(x_0))$.

$\Longrightarrow f(B_\delta(x_0)) \subset B`_\epsilon(f(x_0))$.s

$\Longrightarrow f$ is continuous. $\blacksquare$

Theorem 5.4: A mapping $f: (X, d) \to (X`, d`)$ is continuous iff for each closed set F in $X`$, then $f^{-1}(F)$ is closed set in X.

Proof: $\Longrightarrow$

Suppose that f is continuous and F is closed set in $X`$

To prove $f^{-1}(F)$ is closed in X?

$\because$ F is closed in $X`$

$\Longrightarrow X` - F$ is open in $X`$

$\Longrightarrow f^{-1}(X` - F)$ is open in X.

But $f^{-1}(X` - F) = X - f^{-1}(F)$

$\Longrightarrow f^{-1}(F)$ is a closed set in X.■

$\Longleftarrow$

To prove that f is continuous?

Let U be an open set in $X` \Longrightarrow X` - U$ is closed in $X`$

$\Longrightarrow f^{-1}(X` - U)$ is closed in X. (why ?)

But

$f^{-1}(X` - U) = X - f^{-1}(U)$,

$\Longrightarrow f^{-1}(U)$ is open in X.

$\Longrightarrow f$ is continuous.■

5.1. The Convergence and Continuity (الاستمرارية والتقارب)

Theorem 5.6: A mapping $f:(X,d) \to (X`,d`)$ is continuous at $x_o \in X$ iff for every sequence $\langle x_n \rangle$ in X converges to x_0, then the sequence $\langle f(x_n) \rangle$ in $X`$ converges to $f(x_0)$.

Proof: $\Longrightarrow$

Suppose that f is continuous at $x_o$and $\langle x_n \rangle$ is a sequence in X, such that

$x_n \to x_0$, to prove that $f(x_n) \to f(x_0)$ in X`.

Let U be an open set to X` such that $f(x_0) \in U$.

$\Longrightarrow f^{-1}(U)$ is open in X, and $x_0 \in f^{-1}(U)$.

Since $x_n \to x_0$

$\Longrightarrow f^{-1}(U)$ contains all but a finite number of the term of $\langle x_n \rangle$

$\Longrightarrow U$ contains all but a finite number of the term of $\langle f(x_n) \rangle$

$\Longrightarrow f(x_n) \to f(x_0)$. ■

$\Longleftarrow$

Suppose that if $x_n \to x_0$ in X, then $f(x_n) \to f(x_0)$ in X`

To prove that f is continuous at x_0?.

Suppose that f is not continuous at x_0

$\Longrightarrow \exists\ \epsilon > 0$ such that $\forall n \in N, f(B_{\frac{1}{n}}(x_0)) \not\subseteq B`_{\epsilon}(f(x_0))$

i.e.

. $\forall n \in N, \exists\ x_n \in X$ such that if $d(x_n, x_o) < \frac{1}{n}$

then,

$d`(f(x_n), f(x_0)) \geq \epsilon$

$\Longrightarrow f(x_n) \nrightarrow f(x_0)$, but $x_n \to x_0$, C!

Since $\epsilon > 0 \Longrightarrow \exists\ k \in N$ such that.

$\frac{1}{k} < \epsilon$

$$\Longrightarrow d(x_n, x_0) < \frac{1}{n} < \frac{1}{k} < \epsilon, \forall\, n > k.$$

$\therefore f$ is continuous at x_0.■

Theorem 5.7: Let $f: (X, d) \to (X`, d`)$ and $g: (X`, d`) \to (X'', d'')$ be a mapping such that f is continuous at $x_0 \in X$ and g is continuous at $f(x_0) \in X`$, then $g \circ f$ continuous at $x_0 \in X$.

Proof:

To prove that $g \circ f: (X, d) \to (X'', d'')$ is continuous at $x_0 \in X$.?

Let $\langle x_n \rangle$ be a sequence in X such that

$x_n \to x_0$ and f is continuous at x_0

$\Longrightarrow f(x_n) \longrightarrow f(x_0)$.(why?)

$\because g$ is continuous at $f(x_0)$

$\Longrightarrow g(f(x_n)) \longrightarrow g(f(x_0))$,

$\Longrightarrow (g \circ f)(x_n) \longrightarrow (g \circ f)(x_0)$, (why?)

$\Longrightarrow g \circ f$ is continuous at x_0.■

Examples 5.8: Let $f: R \to R$ be a mapping such that

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Is f continuous mapping at $x = 0$.?

Solution:

The function f is not continuous at $x = 0$. Since

$\frac{1}{n} \to 0,$

but

$f\left(\frac{1}{n}\right) = 1 \Longrightarrow 1 \neq f(0) = 0.$■

Example 5.9: Let $f: [a, b] \to R$ be a mapping such that

$$f(x) = \begin{cases} 1 & \text{if } x \in Q \\ 2 & \text{if } x \notin Q \end{cases}$$

Then, f is not continuous.

Solution:

Let $x_0 \in [a, b]$. If

$x_0 \notin Q \Longrightarrow \exists x_n \in Q$ such that,

$x_n \to x_0$, but $f(x_n) = 1$

$\Longrightarrow 1 \neq f(x_0) = 2$

Also, if $x_0 \in Q$

$\Longrightarrow \exists x_n \in Q^c$ such that

$x_n \to x_0$, but $f(x_n) = 2$

$\Longrightarrow 2 \neq f(x_0) = 1$

$\Longrightarrow f$ is not continuous.■

Definition 5.10: If $f: X \to X`$ is a mapping and $S \subset X$, then $f_S: S \to X`$ is also a mapping such that $f_s(x) = f(x), \forall x \in S$. Therefore, f_s is called the **restriction** of f to S.

Remark 5.10": If f is continuous, then so is f_s, but the converse is not true.

5.2 The Continuous of Real Valued Mapping (استمرارية الدوال ذات القيمة الحقيقية p)

Definition 5. 11: Let X be a metric space , then,

$$C(x) = \{f / f : X \to R \text{ is continuous mapping}\}$$

is the set of all continuous real valued mapping then,

$$C(x) \neq \varphi.$$

If $\exists\, f \in C(x)$ such that

$$f(x) = c, \text{where } c \in R,$$

then,

$$\forall x \in X \text{ is continuous.}$$

Theorem 5.12: If f and g are continuous real valued mapping, then:

1- $f + g$ is continuous such that $(f + g)(x) = f(x) + g(x)$.

2- $f.g$ is continuous such that $(f.g)(x) = f(x).g(x)$.

3- $\forall a \in R, af$ is continuous such that $(af)(x) = af(x)$.

4- If $g(x) \neq 0$, then $\frac{f}{g}$ is continuous such that $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$.

5- $|f|$ is continuous such that $|f|(x) = |f(x)|$.

Proof:

1-To prove that $f+g: X \to R$ is continuous?.

Let $x_0 \in X$ and $\langle x_n \rangle$ be a sequence in X such that

$$x_n \to x_0$$

$\because f$ is continuous

$$\Rightarrow f(x_n) \to f(x_0) \text{ in } R.$$

$\because g$ is continuous

$$\Rightarrow g(x_n) \to g(x_0) \text{ in } R$$

$$\Rightarrow f(x_n) + g(x_n) \to f(x_0) + g(x_0) \text{ in } R$$

$$\Rightarrow (f+g)(x_n) \to (f+g)(x_0)$$

$$\Rightarrow f+g \text{ is continuous at } x_0. \blacksquare$$

Remarks 5.13:

1- From 1and 2 in above theorem $(C(x), +, .)$ is a vector space. (why ?)

2- Any polynomial $p(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$ is continuous. (why ?)

5.3 The Real Mapping on Compact space (الدوال الحقيقية في الفضاء المرصوص)

Definition 5.14 (Bounded Mapping)

A mapping $f: X \to R$ is called bounded if there is $M > 0$ such that

$$|f(x)| \leq M, \forall x \in X.$$

Or

R_f (range of f) is bounded set in R.

Example 5.15:

A mapping $f(0,1) \to R$ such that $f(x) = 2x, \forall x \in (0,1)$ is bounded.

Solution:

Since $\exists\ 2 > 0,$ then $|f(x)| = |2x| \leq 2, \forall x \in (0, 1)$. That means:

$|f(x)| \leq 2, \forall x \in (0, 1)$, with $M = 2.$■

Theorem 5.16: Let X and $X`$ be metric space and $f: X \to X`$ be a continuous mapping if X is compact, then $f(X)$ is compact.

Proof:

Let X be a compact space. To prove that $f(X)$ is compact.

Suppose that $\{V_\alpha \,|\, \alpha \in \Lambda\}$ be any open cover of $f(X), \forall\, x \in X.$

$\Longrightarrow f(X) \subseteq U_{\alpha\in\Lambda} V_\alpha$.

and

$V_\alpha \subseteq_{open} X`, \forall \alpha \in \Lambda$.

$\Longrightarrow f^{-1}\big(f(X)\big) \subseteq f^{-1}(U_{\alpha\in\Lambda} V_\alpha)$.

$\Longrightarrow X \subseteq U_{\alpha\in\Lambda} f^{-1}(V_\alpha)$.

$\because f\ is$ continuous and $V_\alpha \subseteq_{open} X`, \forall \alpha \in \Lambda$.

$\Longrightarrow f^{-1}(V_\alpha) \subseteq_{open} X, \forall \alpha \in \Lambda$.

$\Longrightarrow \{f^{-1}(V_\alpha) \,|\, \alpha \in \Lambda\}$ is an open cover of X.

$\because X$ is compact, then

$\exists\ \{f^{-1}(V_{\alpha_i} \,|\, i = 1,2, \dots, n\}$ is a finite sub cover.

i.e.,

$X \subseteq U_{i=1}^{n} f\left(V_{\alpha_i}\right)$.

$\Longrightarrow f(X) \subseteq U_{i=1}^{n} V_{\alpha_i}$.

$\Longrightarrow f(X)$ is compact.■

Theorem 5.17: A mapping $f: X \to R$ is bounded iff is continuous on a compact space X.

Proof:

$\because f: X \to R$ is continuous and X is compact, then by above theorem $f(X)$ is compact in R.

$\Longleftrightarrow f(X)$ is bounded.

$\Longleftrightarrow \exists\, M > 0$ such that $|f(x)| \leq M, \forall x \in X$.

$\Longleftrightarrow f$ is bounded.■

Remark 5.18: If $f: X \to R$ is continuous and X is not compact, then f is not necessary bounded, consider the following examples:

1-$f(x) = \frac{1}{x}, \forall x \in (0, \infty)$

$\Longrightarrow f$ is continuous and not bounded.

2-$f(x) = 2x, \forall x \in (0,1) \Longrightarrow f$ is continuous and bounded, since

$\exists\, 2 > 0$ such that $|2x| \leq 2, \forall x \in (0,1)$.

Or

$f(x) = f((0,1)) = (0,2)$ is bounded.

$\Longrightarrow f$ is bounded.■

Theorem 5.19: If $f: X \to R$ is a real continuous mapping and X is compact space, then there are $x_0, y_0 \in X$ such that

$$f(y_0) \leq f(x) \leq f(x_0), \forall x \in X.$$

Proof:

$\because f$ is continuous and X is compact.

$\Longrightarrow f(X) = Y$ is compact in R.

$\Longrightarrow f(X) = Y$ is bounded and closed [any compact subset of metric space is closed and bounded].

$\Longrightarrow Y$ has Sup and Inf.

Let $Sup\, Y = M \Longrightarrow M \in Y$, ($Y$ is closed)

$\Longrightarrow \exists\, x_0 \in X$ such that $f(x_0) = M$.

Also, let $m = Inf\ Y \Longrightarrow m \in Y$, ($Y$ is closed)

$\Longrightarrow \exists y_0 \in X$ such that $f(y_o) = m$

$\Longrightarrow \exists\, x_0, y_0 \in X$ such that

$f(y_0) \leq f(x) \leq f(x_0)$, $\forall x \in X$.■