5.7 Differentiation of the Functions (اشتقاق الدوال)

Definition 5.39:

1- Let $D \subseteq R$ and $p \in D$ and let

$$F(x) = \frac{f(x)-f(p)}{x-p}, \text{ (} F \text{ is a function defined on } D/\{p\}\text{)}$$

If $\lim_{x\to p} F(x)$ exists, then f is differentiable at p and

$$\lim_{x\to p} F(x) = \lim_{x\to p} \frac{f(x)-f(p)}{x-p} = f'(p),$$

where $f'(p)$ is the derivative of f at p.

2- If f is differentiable at each point $x \in D$, then we say that f is differentiable on D and $f'(x)$ is denoted the derivative function of f on D.

3- Then, $f'(p)$ is a real number, $f'(x)$ is a function.

Example 5.40:

1- Let $f(x) = c, \forall x \in R$, then,

$$f'(p) = \lim_{x\to p} \frac{f(x)-f(p)}{x-p} = \lim_{x\to p} \frac{c-c}{x-p} = 0$$

$$\Longrightarrow f'(x) = 0, \forall x \in R.$$

2- Assume that

$$f(x) = x, \forall x \in R,$$

then,

$$f'(p) = \lim_{x\to p} \frac{f(x)-f(p)}{x-p} = \lim_{x\to p} \frac{x-p}{x-p} = 1.$$

$$\Longrightarrow f'(x) = 1, \forall x \in R.$$

3- Suppose that

$$f(x) = x^2, \forall x \in R,$$

then,

$$f'(p) = \lim_{x\to p} \frac{f(x)-f(p)}{x-p} = \lim_{x\to p} \frac{x^2-p^2}{x-p}$$

$$= \lim_{x\to p}(x+p) = 2p.$$

Therefore

$$f'(p) = 2p.$$

$$\Longrightarrow f'(x) = 2x, \forall x \in R. \blacksquare$$

Definition 5.41: Let $f: D \to R$ be a function and D be an open interval, a function f is said to be differentiable at $x_0 \in D$, if for any sequence $< x^n >$ in D such that

$$x_n \to x_0, (i.e., x_n \neq x_0),$$

then,

$$\frac{f(x_n)-f(x_0)}{x_n-x_0}$$

converges to the constant number $\alpha = \alpha(x_0)$.*i.e.*

$$\lim_{x_n\to x_0} \frac{f(x_n)-f(x_0)}{x_n-x_0} = \alpha(x_0) = \alpha = f'(x_0) \,,$$

or

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=\alpha(x_0)=f'(x_0).$$

Theorem 5.42: If f is differentiable function at $x_0 \in D$, then f is continuous at x_0.

Proof :

If f is differentiable at $x_0 \in D$, then,

$$f'(x_0)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}.$$

$$\Longrightarrow \lim_{x\to x_0}(x-x_0)f'(x_0)=\lim_{x\to x_0}(f(x)-f(x_0)).$$

$$\Longrightarrow \lim_{x\to x_0}[f(x)-f(x_0)]=0.$$

$$\Longrightarrow \lim_{x\to x_0} f(x)=f(x_0).$$

$\Longrightarrow f$ is continuous at x_0.■

Remark 5.43: The converse of above theorem is not true. Now, consider the following example.

Example 5.44:

Let $f: R \to R$ be a function such that

$f(x)=|x| \ \forall x \in R,$

$\Longrightarrow f$ is continuous on R (why..?)

$\Longrightarrow f$ is continuous at $x_0=0$. (why..?)

But f is not differentiable at $x_0=0$, (why..?)

Since

$$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$$

$$= \lim_{x\to 0} \frac{|x|-0}{x-0} = \lim_{x\to 0} \frac{|x|}{x}$$

$$= \lim_{x\to 0} \begin{cases} \frac{x}{x}, & x > 0 \\ \frac{-x}{x}, & x < 0 \end{cases} .$$

$\Longrightarrow f'(0)$ does not exist.

$\Longrightarrow f$ is not differentiable at $x_0 = 0$.■

Theorem 5.46: If $f: D \to R$ and $g: D \to R$ are two differentiable functions at $x_0 \in D$ and $c \in R$, then:

1- $f + g$ is differentiable at x_0, and

$$(f+g)'(x_0) = f'(x_0) + g'(x_0) .$$

2- $f - g$ is differentiable at x_0, and

$$(f-g)'(x_0) = f'(x_0) - g'(x_0) .$$

3- $f.g$ is differentiable at x_0, and

$$(f.g)'(x_0) = f'(x_0).g'(x_0) .$$

4- $C.f$ is differentiable at x_0, and

$$(C.f)^{'(x_0)} = C.f'(x_0).$$

5- If $g(x_0) \neq 0$, then $\frac{f}{g}$ is differentiable at x_0, and

$$\left(\frac{f}{g}\right)'(x_0) = \frac{g(x_0)f'(x_0) - f(x_0)g'(x_0)}{[g(x_0)]^2}$$

6- Also, $\frac{1}{g}$ is differentiable at x_0, and

$$\left(\frac{1}{g}\right)'(x_0) = \frac{-g\prime x_0)}{[g(x_0)]^2}.$$

5.7.1 The Chain Rule (قاعدة السلسلة)

Theorem 5.47: Let J and L be two open intervals in R and $f: J \to R$ and $g: L \to R$ be two functions such that $f(J) \subset L$. If f is differentiable at $x_0 \in J$ and g is differentiable at $f(x_0) \in L$, then $g \circ f$ is differentiable at $x_0 \in J$ and then,

$$(g \circ f)'(x_0) = g'(f(x_0). f'(x_0)$$

Proof:

Let $h = g \circ f: J \to R, x_0 \in J.$

$$h'(x_0) = \lim_{x \to x_0} \frac{(g \circ f)(x) - (g \circ f)(x_0)}{x - x_0}.$$

$$h'(x_0) = \lim_{x \to x_0} \frac{g\big(f(x)\big) - g\big(f(x_0)\big)}{x - x_0} . \frac{f(x) - f(x_0)}{f(x) - f(x_0)}.$$

$$h'(x_0) = \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} . \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

$$= g'\big(f(x_0)\big). f'(x_0).$$

$$\Rightarrow (g \circ f)'(x_0) = g'(f(x_0). f'(x_0). \blacksquare$$

Theorem 5.48 (The Invers Function Theorem مبرهنة الدالة العكسية)

Let $f: D \to R$ be a one-to-one function and differentiable at $x_0 \in D$ such that $f'(x_0) \neq 0$, then $f^{-1}: f(D) \to D$ is differentiable at $f(x_0)$ and

$$(f^{-1})\big(f(x_0)\big) = \frac{\mathbf{1}}{\boldsymbol{f'(x_0)}}.$$

Proof: Check…?

5.7.2 Mean Value Theorem (مبرهنة القيمة الوسطى)

Theorem 5.49: Let $f: D \to R$ be a differentiable function at $p \in D$. If $f'(p) > 0$, then is $\epsilon > 0$ such that

$$f(x) < f(p), \forall x < p \text{ in } N(x,p)$$

and

$$f(x) > f(p), \forall x > p \text{ in } N(p,\epsilon).$$

Proof:

If $f: D \to R$ be a differentiable function at $p \in D$, then,

$$f'(p) = \lim_{x \to p} \frac{f(x)-f(p)}{x-p} > 0 \Longrightarrow \exists\, \epsilon > 0,$$

such that

$$\frac{f(x)-f(p)}{x-p} > 0, \forall x \in N(p,\epsilon).$$

$$\Longrightarrow f(x) - f(p) \text{ and } x - p, \text{ have the same sign.}$$

If

$$x > p \Longrightarrow f(x) > f(p),$$

and if

$$x < p \Longrightarrow f(x) < f(p), \forall x \in N(p,\epsilon).$$

i. e.

$$\Longrightarrow f \text{ is increasing on } N(p,\epsilon). \blacksquare$$

Theorem 5.50: Let $f: D \to R$ be a differentiable function at $p \in D$.

If $f'(p) < 0$, then is $\epsilon > 0$ such that

$$f(x) < f(p), \forall x > p \text{ in } N(p, \epsilon)$$

and

$$f(x) > f(p), \forall x < p \text{ in } N(x, p).$$

Proof:

Suppose that $f: D \to R$ be a differentiable function at $p \in D$, then,

$$f'(p) = \lim_{x \to p} \frac{f(x) - f(p)}{x - p} < 0$$

$\Longrightarrow \exists \epsilon > 0$ such that

$$\frac{f(x) - f(p)}{x - p} < 0 \ \forall x \in N(p, \epsilon).$$

$\Longrightarrow f(x) - f(p)$ and $x - p$ have the same sign.

If $x > p \Longrightarrow f(x) < f(p)$.

If $x < p \Longrightarrow f(x) > f(p), \forall x \in N(p, \epsilon)$.

i. e. $\Longrightarrow f$ is decreasing on $N(p, \epsilon)$.■

5.7.3 A Local Maximum and Minimum Point (نقطة النهاية العظمى والصغرى المحلية)

Definition 5.51: A point p is called a *local maximum* point (**l.m.p.**) of f if there is a $N(p, \epsilon)$ such that

$$f(p) \geq f(x), \forall x \in N(p, \epsilon) .$$

Definition 5.52 (A local minimum point):

A point p is called a local *minimum point* (**l. mi. p**.) of f if there is a $N(p,\epsilon)$ such that

$$f(p) \leq f(x)\ \forall x \in N(p,\epsilon).$$

Examples 5.53:

1- Let $f: R \to R$ such that

$$f(x) = 3x^4 - 4x^3 - 12x^2, \text{ then,}$$

$$f'(x) = 12x^3 - 12x^2 - 24x \Longrightarrow f'(x) = 0.$$

$$\Longrightarrow 12x\,(x^2 - x - 2) = 0 \Longrightarrow 12x\,(x+1)(x-2) = 0.$$

$$\Longrightarrow x = 0, x = -1, x = 2.$$

Since $f(0) \geq f(x), \forall x \in N(0,\epsilon) \Longrightarrow 0$ is **l. m. p**. of f.

If $f(-1) \leq f(x), \forall x \in N(-1,\epsilon) \Longrightarrow -1$ is **l. mi. p.** of f.

Now if, $f(2) \leq f(x), \forall x \in N(2,\epsilon) \Longrightarrow 2$ is **l. mi. p**. of f.

2- Let $f: R^+ \to R$ such that

$$f(x) = \sin(1/x) \text{ then,}$$

$$f'(x) = \frac{-1}{x^2}\cos\left(\frac{1}{x}\right).$$

$$\Longrightarrow f'(x) = 0 \Longrightarrow \frac{-1}{x^2}\cos\left(\frac{1}{x}\right) = 0.$$

therefore

$x = \frac{2}{(4n-1)\pi}$ is **l. mi. p** of f $(n \in N)$, and

$$x = \frac{2}{(4n+1)\pi} \text{ is } \mathbf{l.\ m.\ p} \text{ of } f \ (n \in N).■$$

Theorem 5.54: Let $f: D \to R$ be a differentiable function at $p \in D$, if p is a local maximum point or a local minimum point, then

$$f'(p) = 0.$$

Proof:

Let $f: D \to R$ be a differentiable function at $p \in D$, if p is a local maximum point or a local minimum point.

If $f'(p) > 0 \Longrightarrow \exists$ a $N(p, \epsilon)$ such that f is increasing function p is not a l.mi.p and not a l. m. p, **C!.**

If $f'(p) < 0 \Longrightarrow \exists a, N(p, \epsilon)$ $s$uch that f is decreasing function p is not a l. mi. p and not a l.m.p., **C!.**

Then,

$$f'(p) = 0.■$$

Remark 5.55: The converse of above theorem is not true. For this purpose, consider the following example:

Example 5.56:

Let $f(x) = x^3$

$$\Longrightarrow f'^{(x)} = 3x^2$$

$$\Longrightarrow f'(x) = 0.$$

$$\Longrightarrow 3x^2 = 0$$

$\Longrightarrow x = 0 \Longrightarrow f'(0) = 0.$

But

$x = 0$ is not a l. m. p or a l. mi. p of f.■

5.7. 3 Rolls Theorem (مبرهنة رول)

Theorem 5.57 : Let f be a continuous function on $[a, b]$ and differentiable on (a, b) if

$f(a) = f(b) = 0$, then is a point $c \in (a, b)$

such that

$f'(c) = 0.$

Proof:

Case (1):

If f is constant mapping

$\Longrightarrow f'(x) = 0, \ \forall x \in (a, b)$

$\Longrightarrow \exists c \in (a, b)$ such that $f'(c) = 0.$

Case (2):

If f is not constant mapping.

Since f is continuous on compact set $[a, b]$, (why..?)

then,

$\Longrightarrow \exists x_0, y_0$ $[a, b]$ such that

x_0 is a $l.m.p$ and y_0 is a $l.mi.p$ such that

$f(y_0) < f(x) < f(x_0), \forall x \in [a, b]$.

If $f(x_0) = f(y_0) \Longrightarrow f$ is constant C!.

$\Longrightarrow f(x_0) \neq f(y_0) \Longrightarrow x_0 \neq y_0$

$\Longrightarrow$ at least one of the point x_0 or $y_0 \neq a$ or b $(f(a) = f(b))$

$\Longrightarrow$ either $x_0 \in (a, b)$ or $y_0 \in (a, b)$

$\Longrightarrow$ either $f'(x_0) = 0$ or $f'(y_0) = 0$ (if p is l.m.p or l.mi.p)

$\Longrightarrow f'(p) = 0 \Longrightarrow \exists c \in (a, b)$ such that $f'(c) = 0$.■

Example 5.58:

1- $f(x) = 3x - x^3, \forall x \in [-\sqrt{3}, \sqrt{3}]$.

Solution 1:

$\because f$ is continuous on $[-\sqrt{3}, \sqrt{3}]$, differentiable on $(-\sqrt{3}, \sqrt{3})$, $f(\sqrt{3}) = f(-\sqrt{3}) = 0$.

Then, by Rolle's theorem $\exists c \in (-\sqrt{3}, \sqrt{3})$ such that $f'(c) = 0$.

$\because f'(x) = 3 - 3x^2 \Longrightarrow f'(c) = 3 - 3c^2$.

$\Longrightarrow f'(c) = 0 \Longrightarrow 3 - 3c^2 = 0$.

$\Longrightarrow c^2 = 1 \Longrightarrow c = \pm 1 \in (-\sqrt{3}, \sqrt{3})$.

2- $f(x) = \sqrt{1 - x^2}$ $\forall x \in [-1,1]$.

Solution 2: Check.■