1-5 Examples

If R is a set of functions f : R # ⟶ R # (R , +, •) is a ring

define (+ and .) (f + g) (a) = f (a) + g(a) , (f . g) (a) = f (a) . g (a) . ∀ a ∈ R #

Solution: ❶ ∀ f, g ∈ R. (f + g)(a) = f(a) + g(a) ∈ R . ∀ a ∈ R (close)

❷ ∀ f, g, h ∈ R . (f + g) + h)(a) = (f + g)(a) + h(a) = (f(a) + g(a)) + h(a) = f(a) + (g(a) + h(a)) = f(a) + ((g + h) (a)) (associative)

❸ Ker f is the identity , hence (f + ker f) (a) = f(a) + ker f(a) = f(a) + 0 = f(a). So (ker f + f)(a) = f(a) .

❹ ∀ f ∈ R ∃ -f ∈ R ∋ (f+(-f)) (a) = f(a) + (-f (a)) = f (a) − f (a) So (-f + f)(a) = - f(a) + f(a) = 0 (inverse)

❺ ∀ f, g ∈ R, (f+g) (a) = f(a)+g(a) = g(a) +f (a) = (g+f) (a) (commutatin) Thus (R , +) is a belian group.

❻ ∀ f , g ∈ R . (f · g) (a) = f(a). g(a) ∈ R (closed under)

❼ ∀ f , g , h ∈ R. ((f. g) (a)) ∙ h (a) = (f (a). g(a)) ∙h(a) = f(a) . (g(a) ∙ h(a)) = f(a). ((g. h) (a)) (associative ander.) Hence (R , .) is a semi-group.

❽ ∀ f, g , h ∈ R . (f . (g + h))(a) = f(a). (g + h) (a) = f (a). (g(a) + h(a)) = f(a) . g(a) + f (a) . h(a) (left distripputire) 6 ((g + h). f) (a) = ((g + h) (a)) ∙ f(a) = (g(a) +h(a)) . f(a) g(a). f (a) + h (a). f (a) (Right distripputive) . Therefore (R , +, ∙) is a ring .

❾ ∀ f , g ∈ R . (f . g)(a) = f(a) . g(a) = g(a) . f(a) = (g. f)(a) (commutativ)

❿ ∀ f ∈ R ∃ I ∈ R ∋ (f. i)(a) = f (a). i (a) = f(a) . i = f(a) So (i . f) (a) = f(a) .

[where i(a) = 1 ∀ a ∈ R #] (identity)

Thus (R , +, ∙) is a commutative ring with identity.